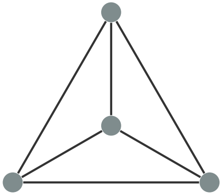
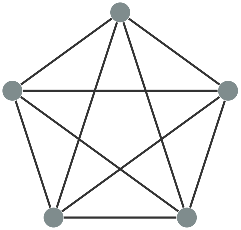
On Planar Graphs of Uniform Polynomial Growth

Farzam Ebrahimnejad

joint work with James R. Lee

Planarity





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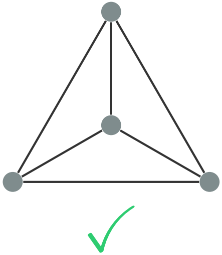

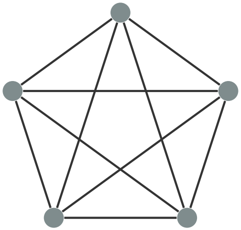

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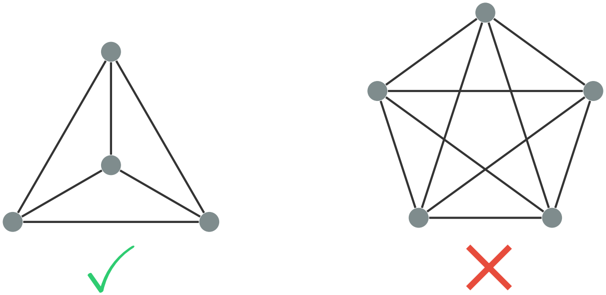

Volume Growth

For a graph G, $v \in V(G)$ and $R \geq 0$ we define

$$B_G(v,R) = \{u : d_G(v,u) \leq R\},$$

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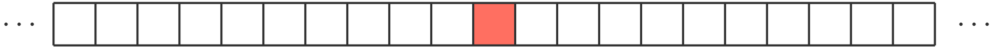
- We are interested in how $|B(v, R)|$ grows as a function of R.

Volume Growth in $\mathbb{Z}$



$$|B(v,0)| = 1$$

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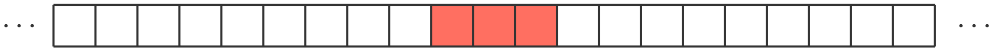
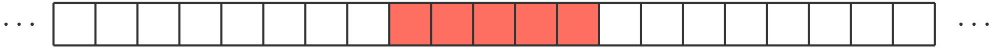


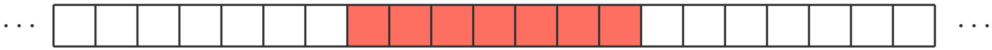
$$|B(v,1)| = 3$$

Volume Growth in $\mathbb{Z}$



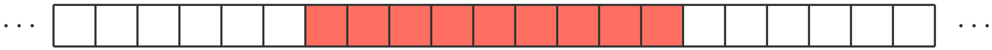
$$|B(v,R)| = 2R + 1$$

Volume Growth in $\mathbb{Z}$



$$
\begin{aligned}
|B(v,R)| &= 2R+1 \\
&= \Theta(R).
\end{aligned}
$$

Volume Growth in $\mathbb{Z}$



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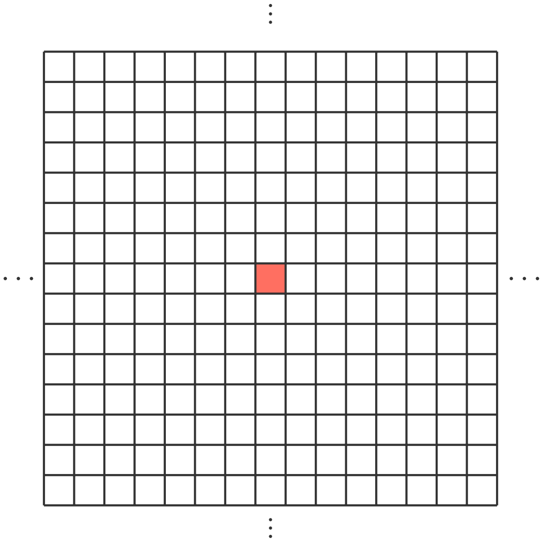

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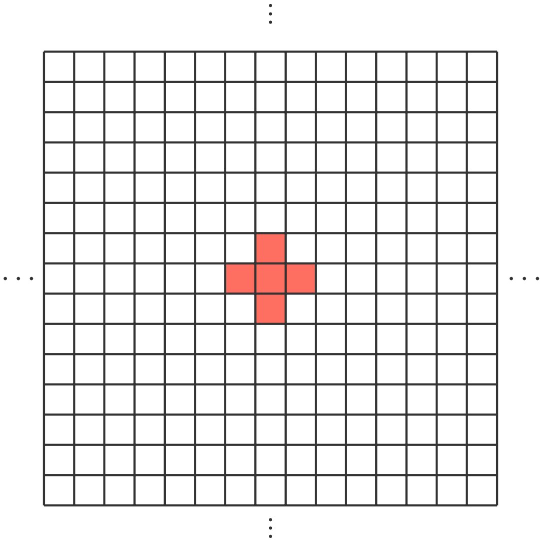
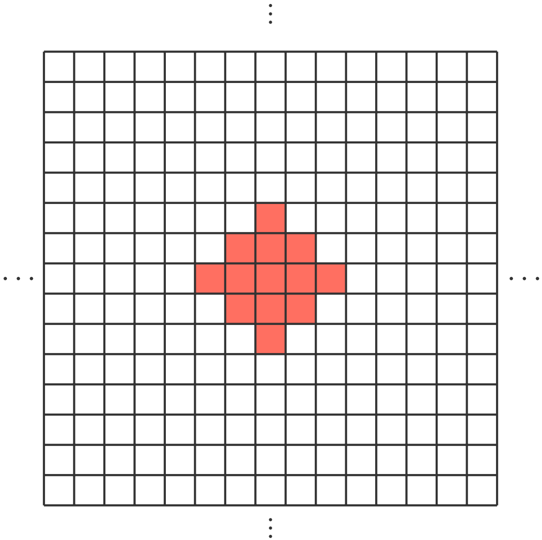


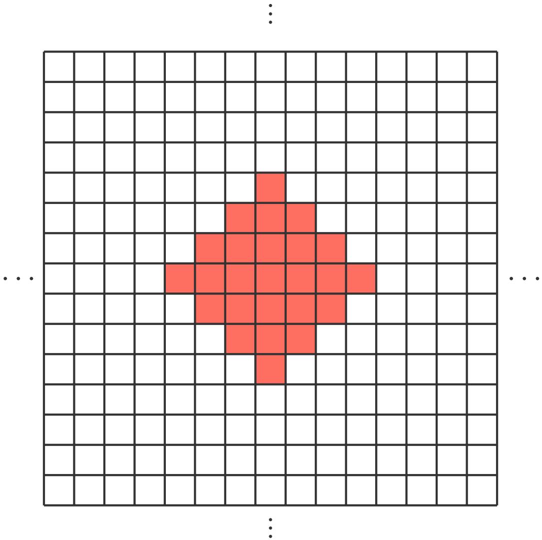
$|B(v,1)| = 5$

Volume Growth in $\mathbb{Z}^2$



$|B(v,2)| = 13$

Volume Growth in $\mathbb{Z}^2$



$$|B(v,R)| = 2R^2 - 2R + 1$$

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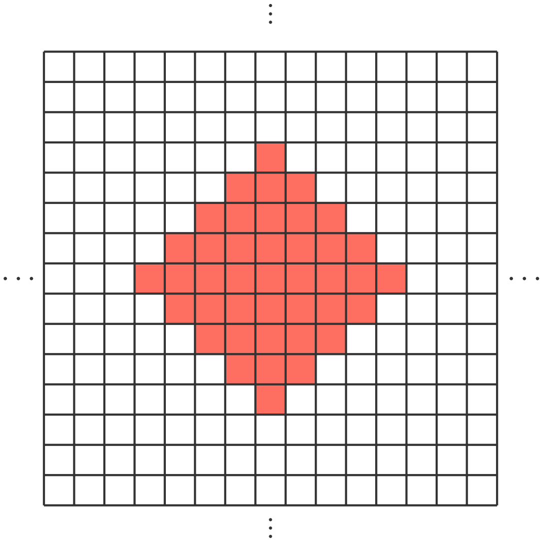

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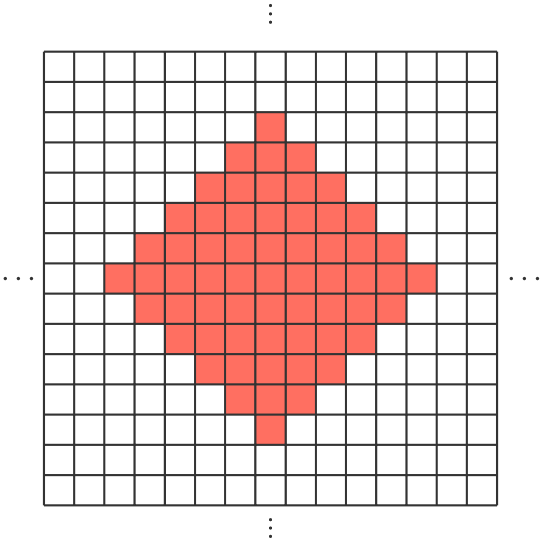

$$|B(v,R)| = 2R^2 - 2R + 1 = \Theta(R^2).$$

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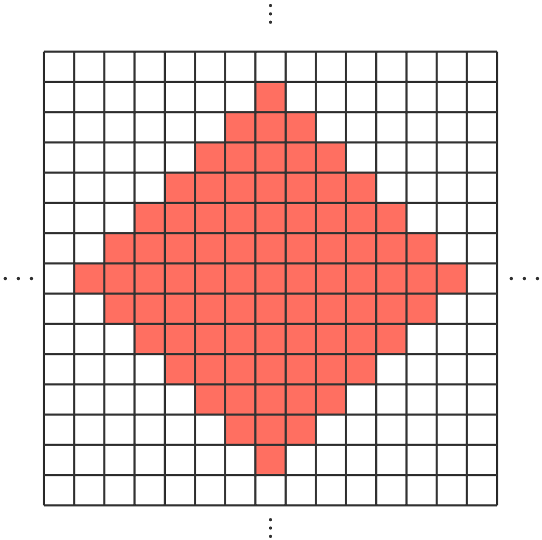

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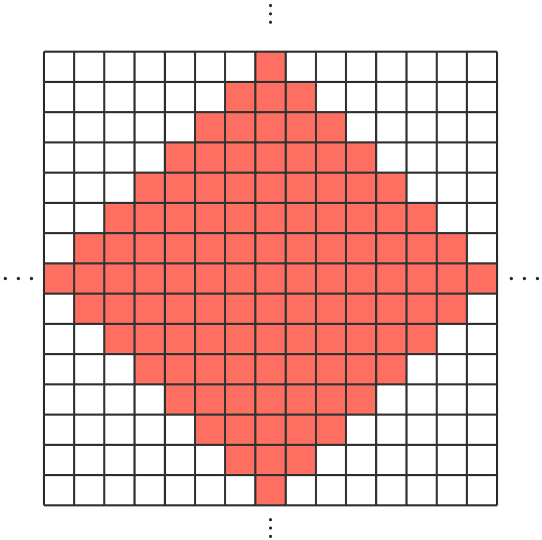

$|B(v,R)|$

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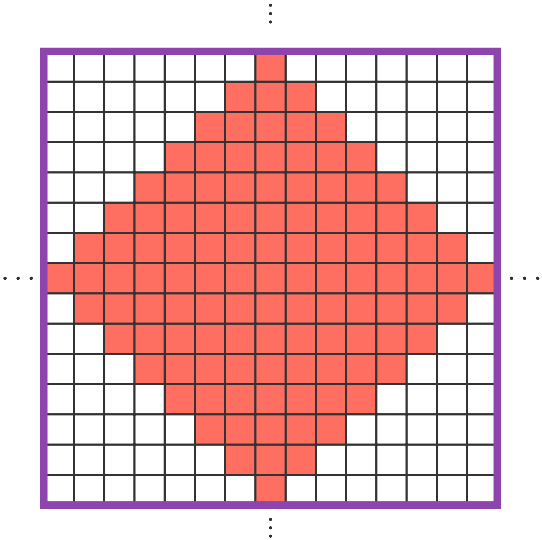

$$|B(v,R)| \leq (2R+1)^2$$

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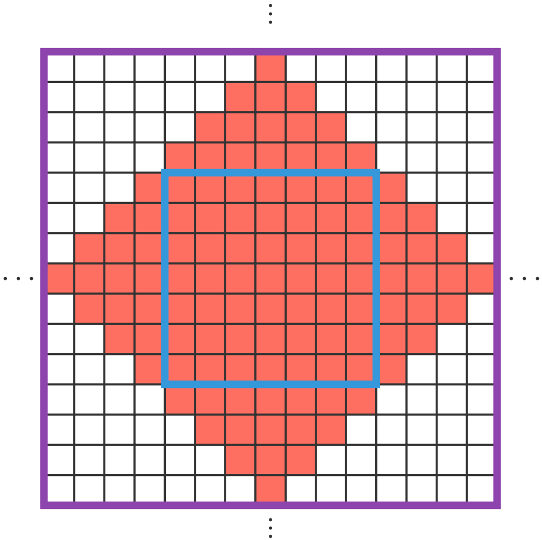
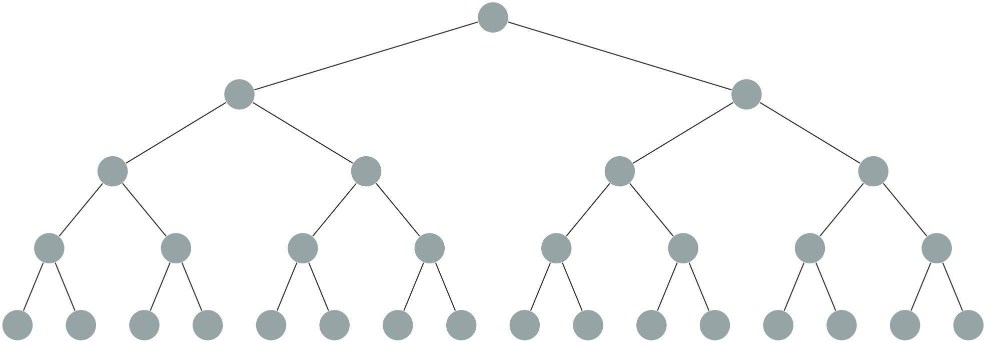


$$(R+1)^2 \leq |B(v,R)| \leq (2R+1)^2$$

Volume Growth in the Binary Tree



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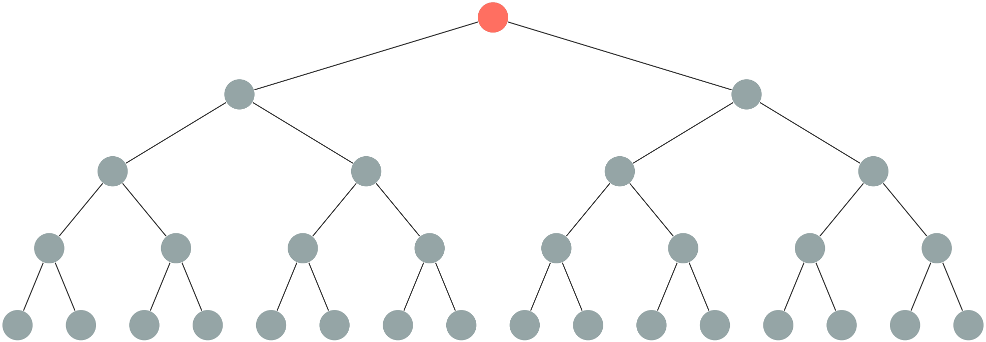

$|B(v, 0)| = 1$

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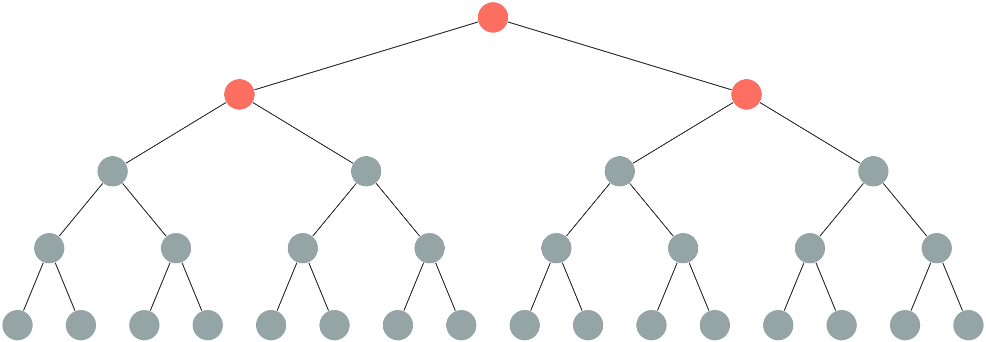

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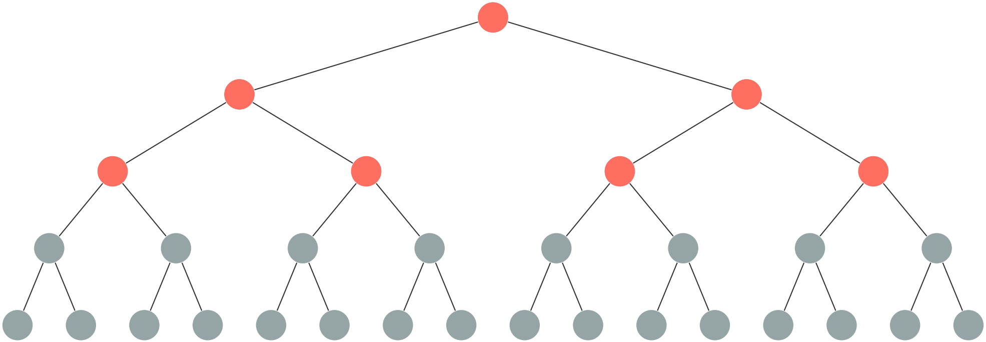

$$|B(v,2)| = 7$$

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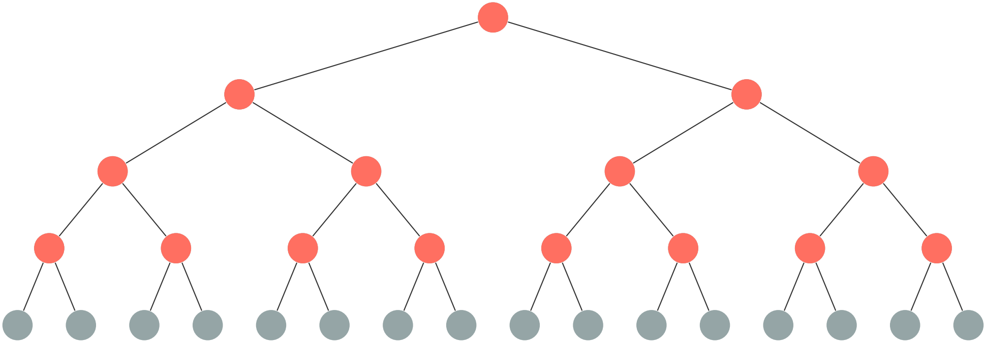

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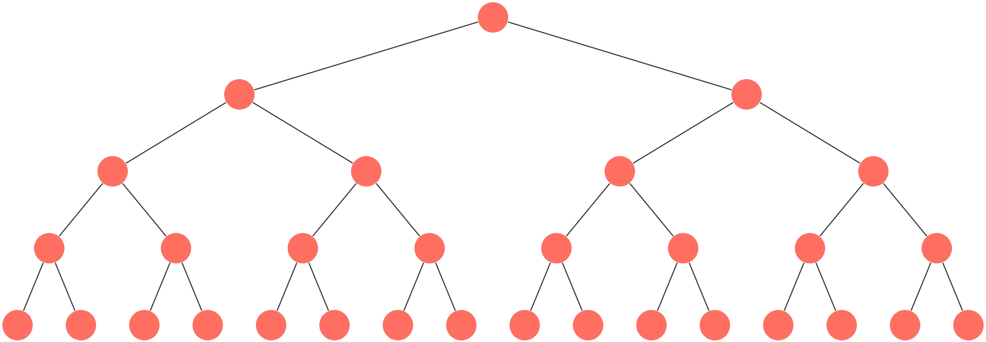

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Uniform Volume Growth

We say a graph G has *uniform volume growth* $f(R)$ if for all $v \in V(G)$ and $R \geq 0$ we have

$$c f(R) \leq |B(v,R)| \leq C f(R)$$

for some constants $0 < c \leq C$.

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Examples

- The *d-dimensional grid* has R^d uniform growth, which is polynomial in R.
- The *infinite binary tree* has 2^R uniform growth, which is exponential in R.
- Not all graphs have uniform volume growth.

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Planar Graphs of Uniform Polynomial Growth > 2

Theorem [Babai '97]

If G is a *vertex-transitive* planar graph of uniform growth, then the growth is either linear, quadratic, or exponential.

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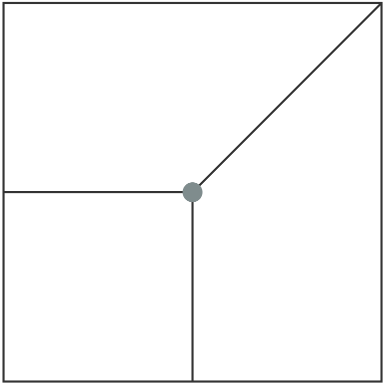
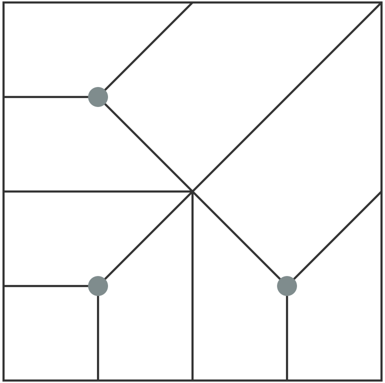
Theorem [Benjamini - Schramm '01]

For all $\alpha \geq 1$, there exists an (infinite) planar graph of uniform growth R^{α}.

A Construction by Benjamini-Schramm

$Q_1 =$ 

A Construction by Benjamini-Schramm

$Q_2 =$ 

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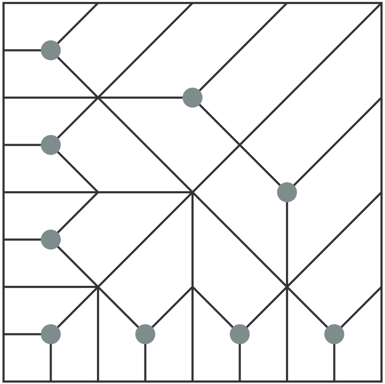
$Q_3 =$ 

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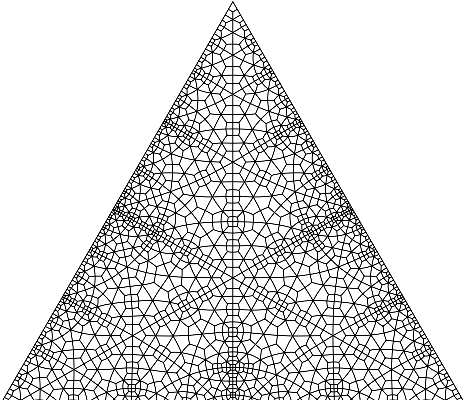
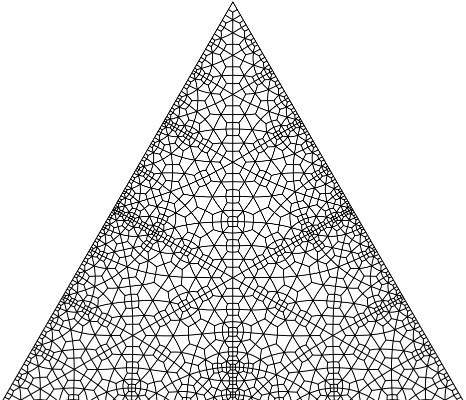


A Construction by Benjamini-Schramm

Theorem [Benjamini - Schramm '01]

The sequence of random rooted graphs $(Q_1, \rho_1), (Q_2, \rho_2), \ldots$, where ρ_i is chosen uniformly among the vertices of Q_i, has a subsequential limit (Q_∞, ρ).



Uniform Infinite Planar Triangulation (UIPT)

Defined as the distributional limit of a random uniform triangulation of the 2-sphere [Angel - Schramm '03].

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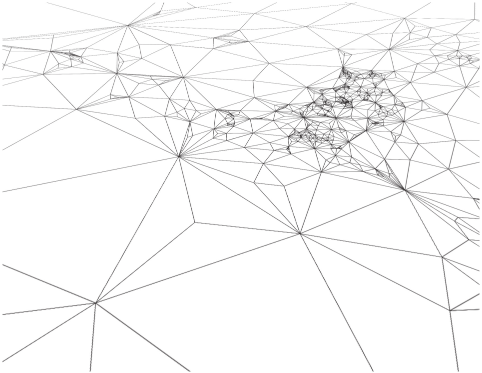

Drawn by Igor Kortchemski

Uniform Infinite Planar Triangulation (UIPT)

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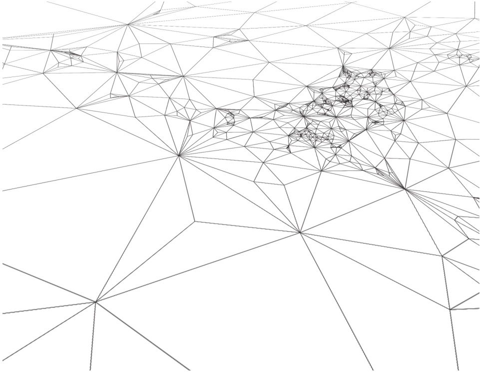
- Balls of radius R in the UIPT almost surely have volume $R^{4+o(1)}$ [Angel '03].



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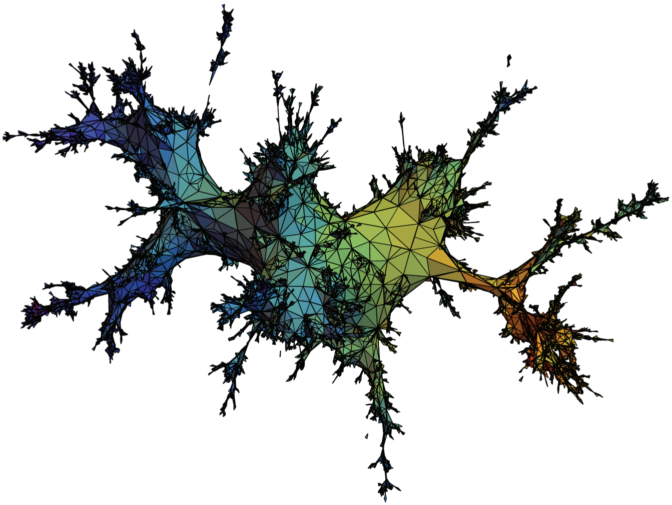

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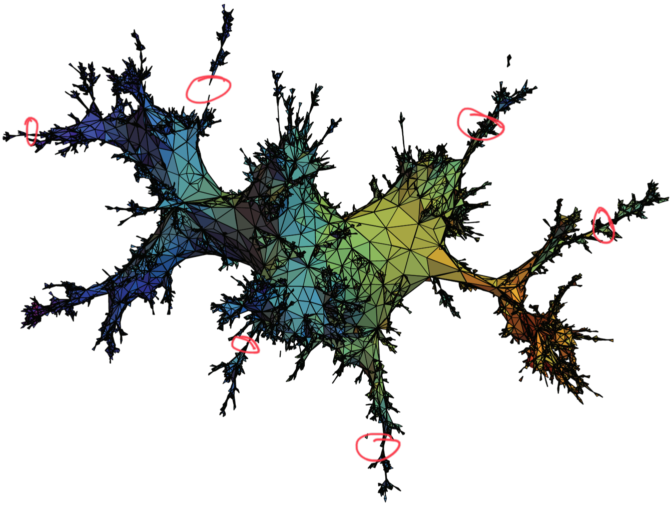

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Problem 1 [Benjamini - Papasoglu '10]

Is there a planar graph of uniform polynomial growth $\alpha > 2$ in which all complements of balls are connected?

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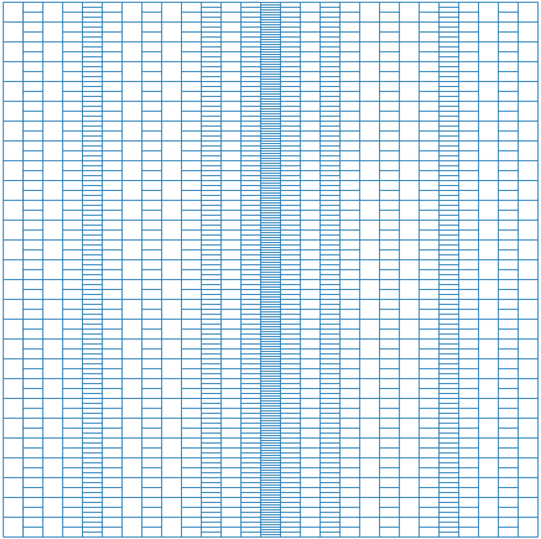
Theorem [E - Lee '21]

For all $\alpha > 2$, there exists a (unimodular) planar graph of uniform growth R^α in which the complements of all balls are connected.

Sneak Peek



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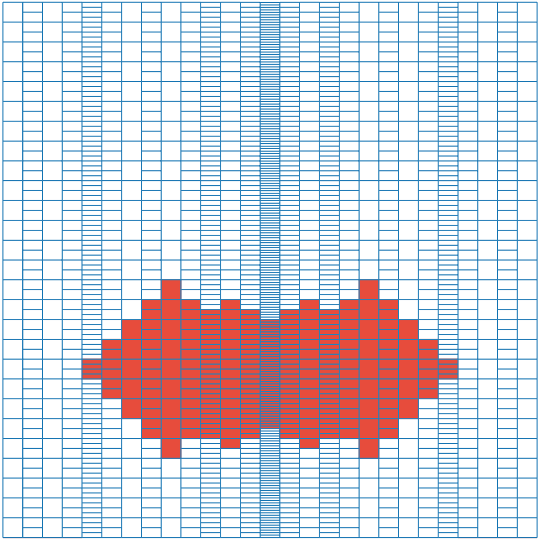

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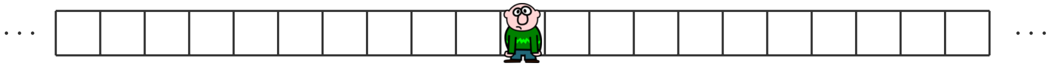
Up next . . .

- ☐ Random walks.
- ☐ Effective resistances.
- ☐ Our construction(s).

Random Walks

The Random Walk on $\mathbb{Z}$



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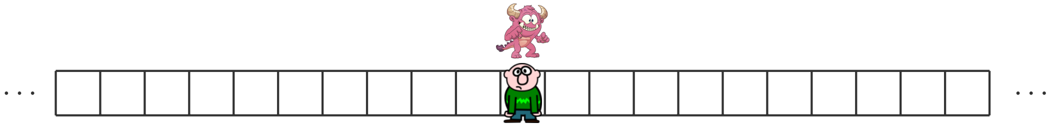

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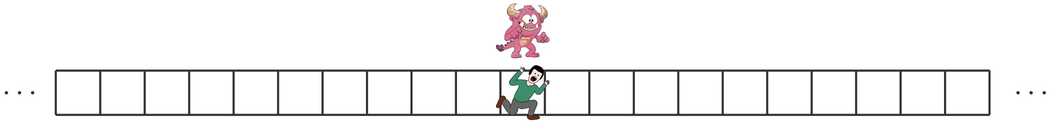

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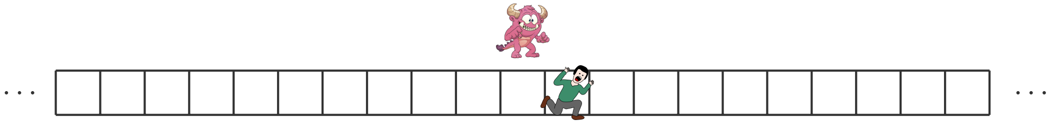

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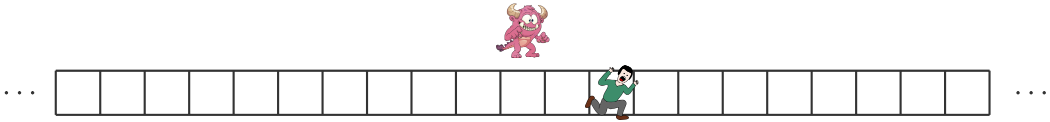

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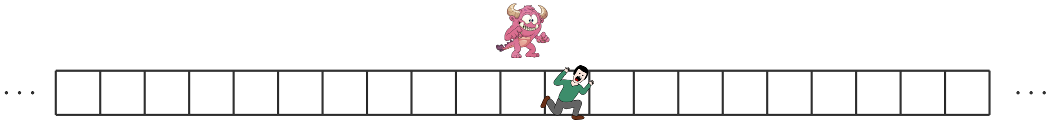

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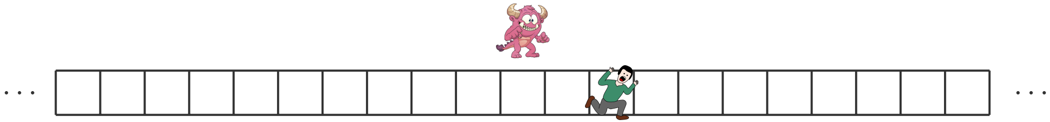

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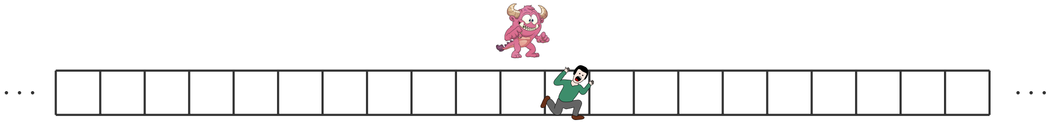

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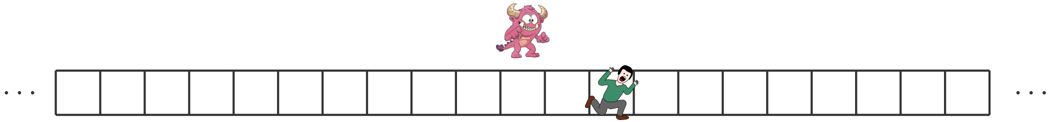

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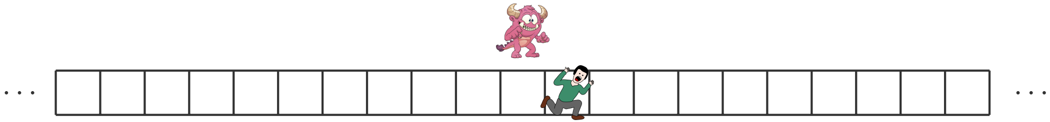

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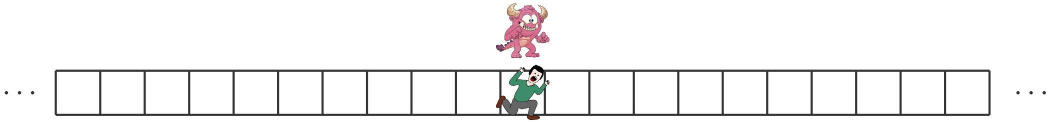

Recurrence vs. Transience

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  - This includes the Benjamini - Schramm constructions and the UIPT.

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Effective Resistance

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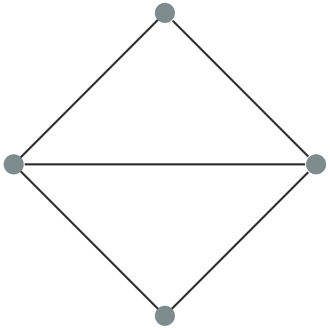

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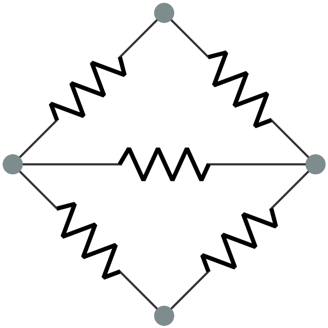

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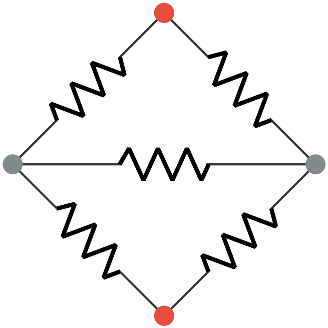

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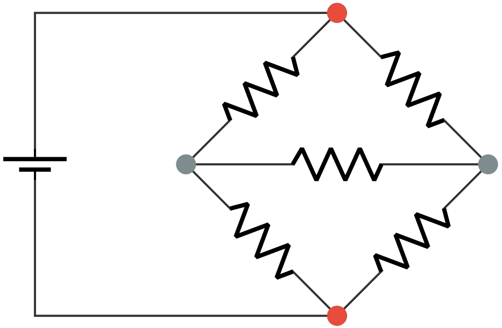

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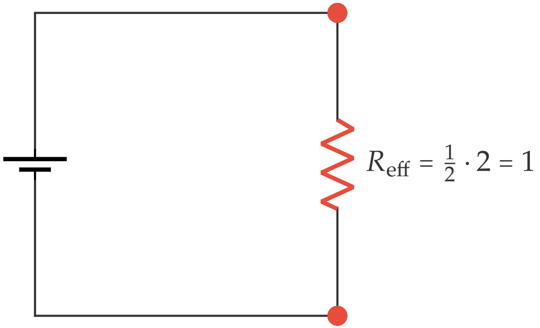

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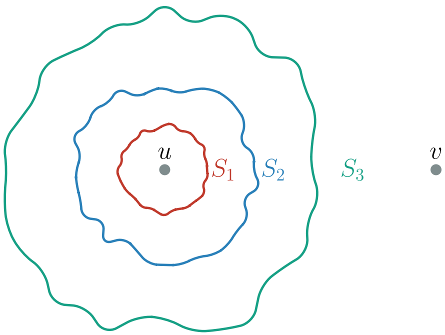
Theorem [Doyle - Snell '84]

$G = (V, E)$ is recurrent if and only if $R_{\text{eff}}(v, \infty) = \infty$ for $v \in V$. Where

$$R_{\text{eff}}(v, \infty) = \lim_{R \to \infty} R_{\text{eff}}\left(v, V \setminus B(v, R)\right).$$

How to Bound the Effective Resistance?



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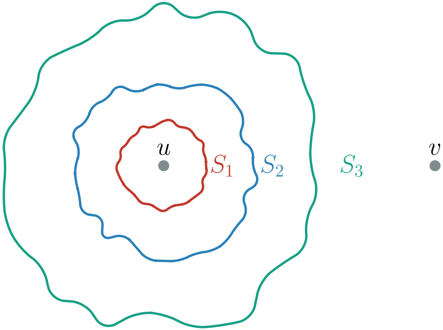
$$R_{\text{eff}}(u,v) \gtrsim \frac{1}{|S_1|} + \frac{1}{|S_2|} + \cdots + \frac{1}{|S_n|}$$



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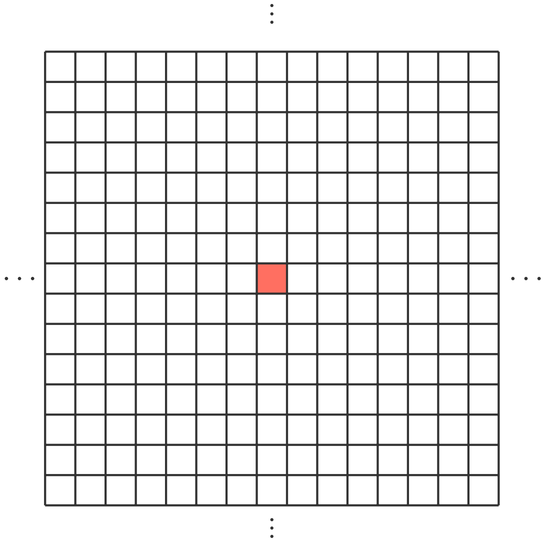

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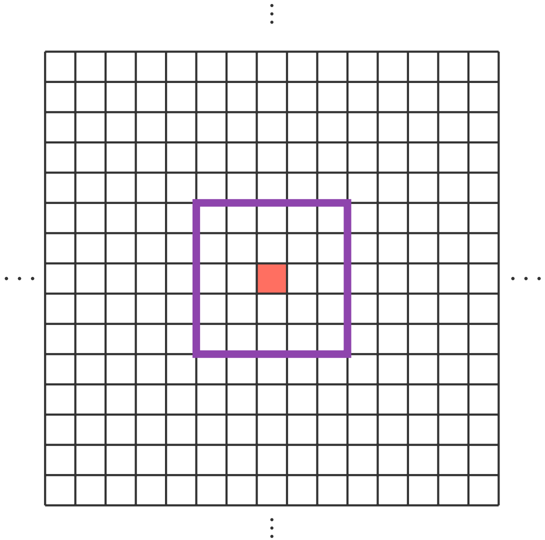

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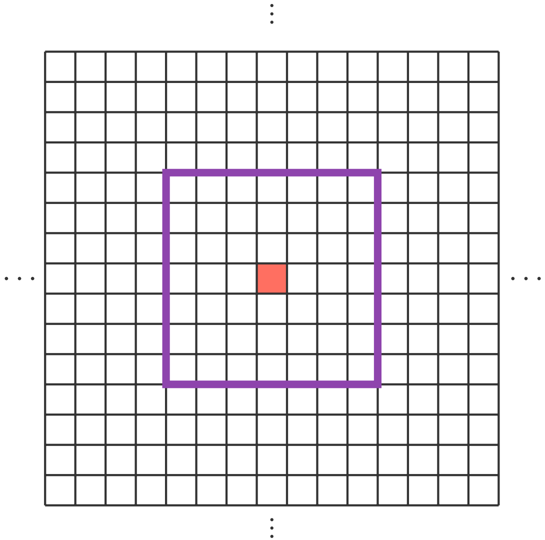

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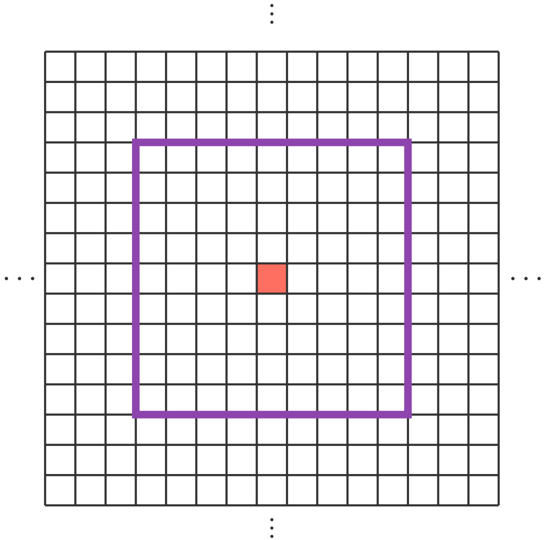

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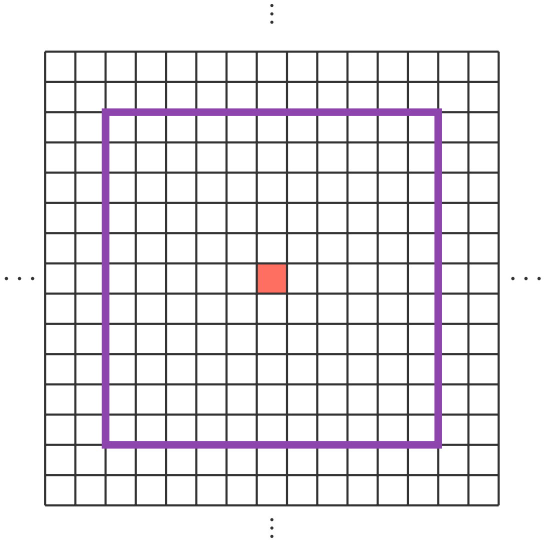

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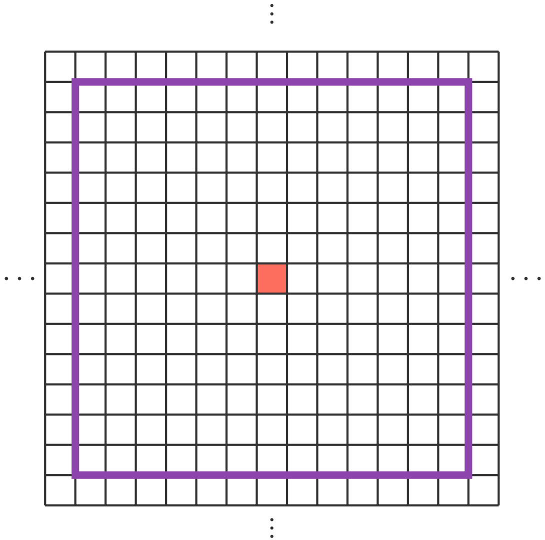

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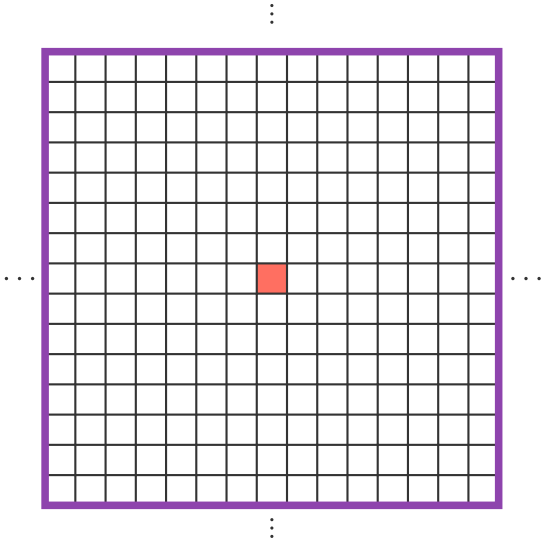

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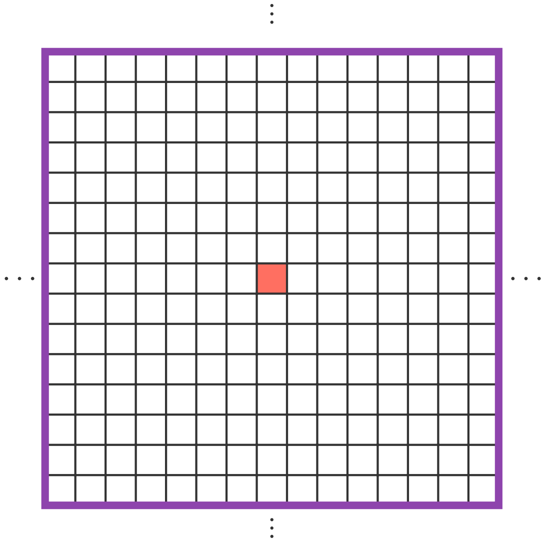

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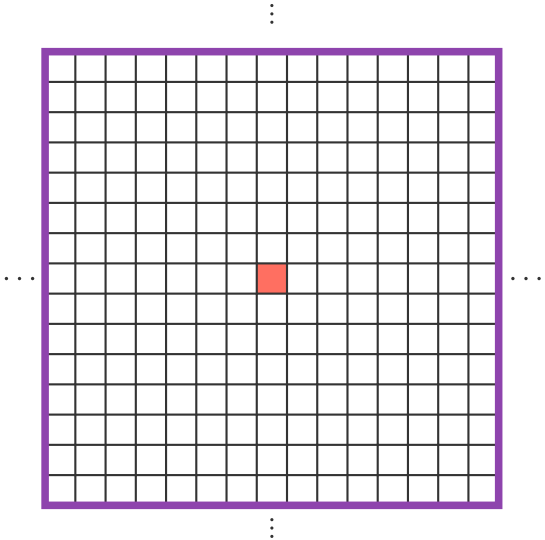

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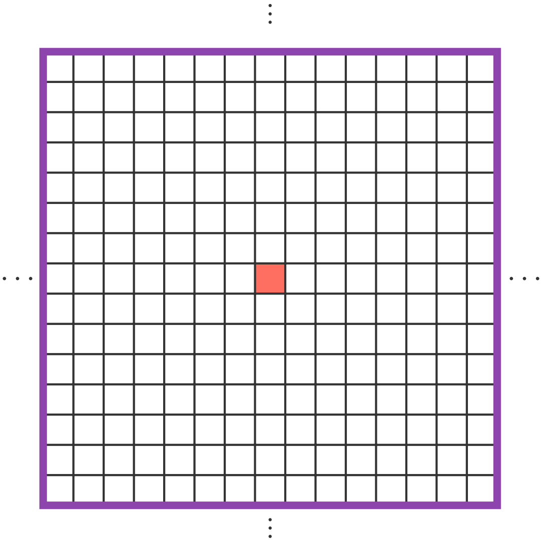

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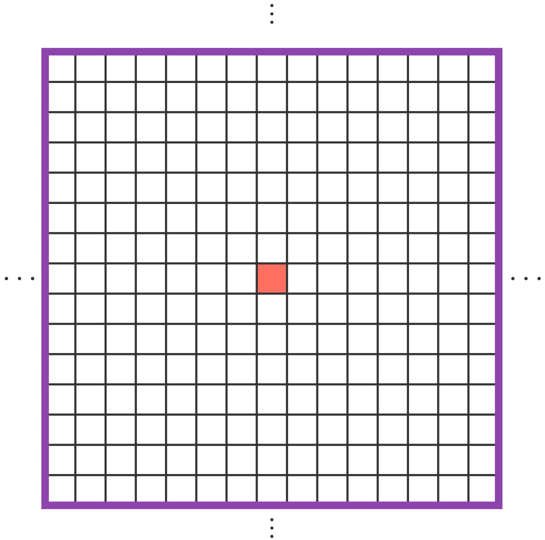

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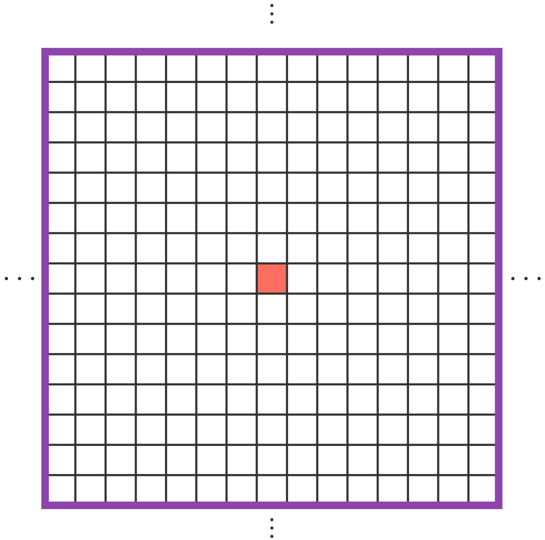

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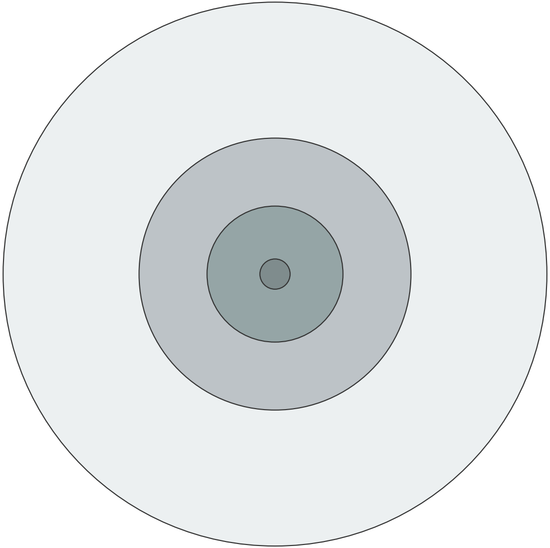

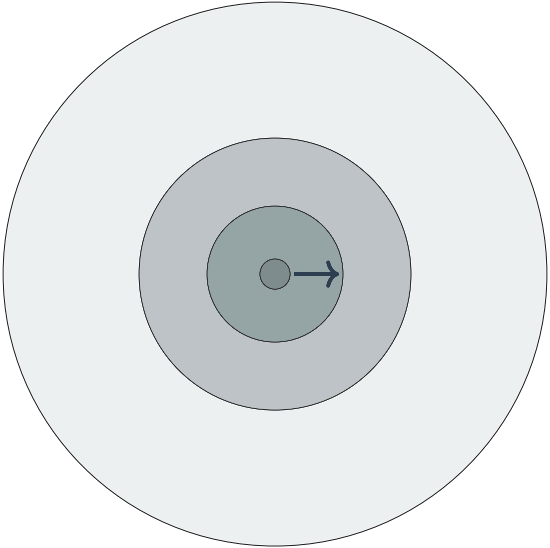

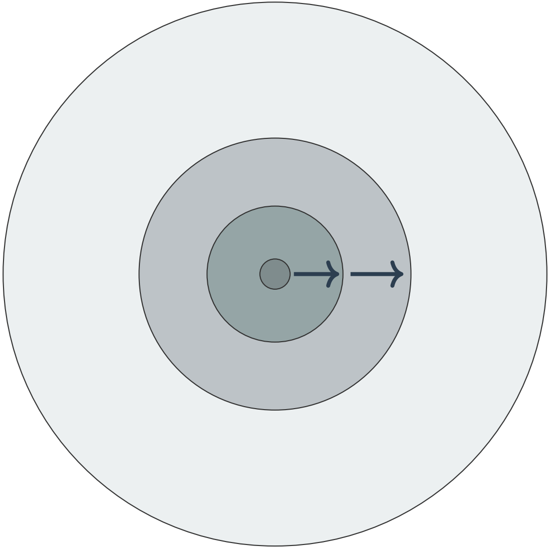

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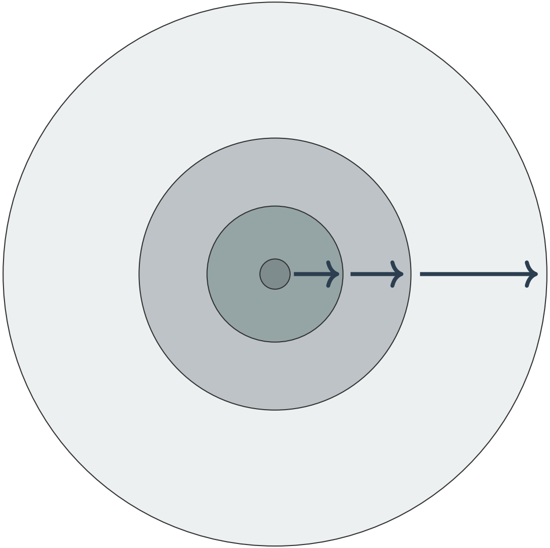

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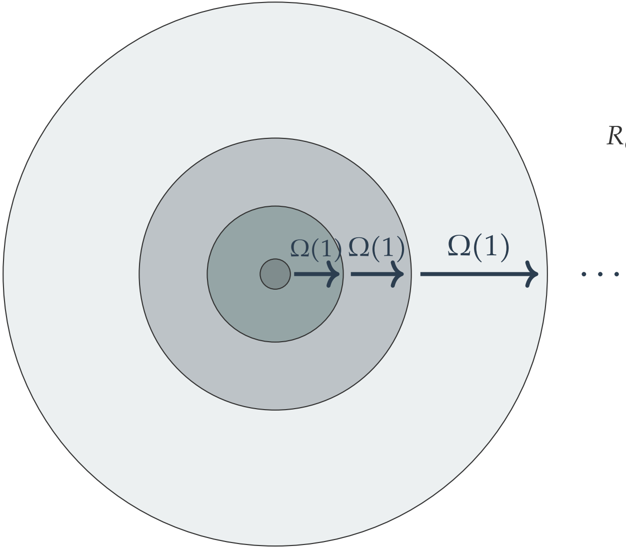

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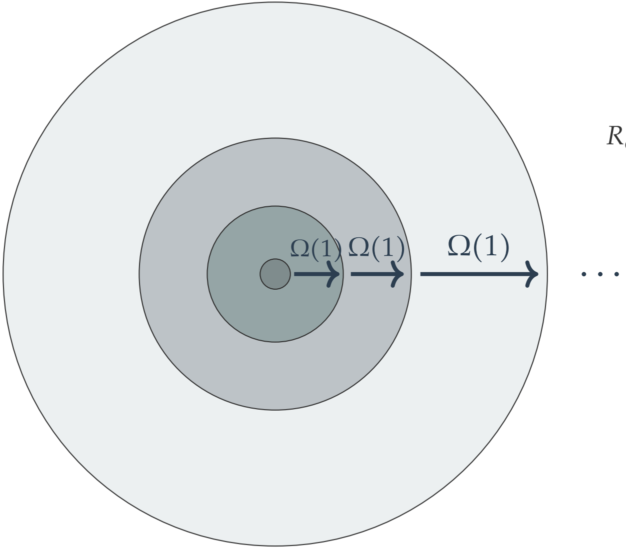

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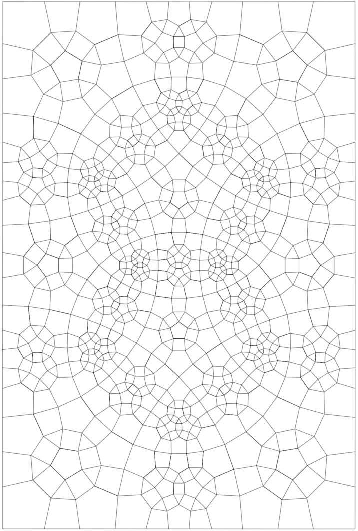

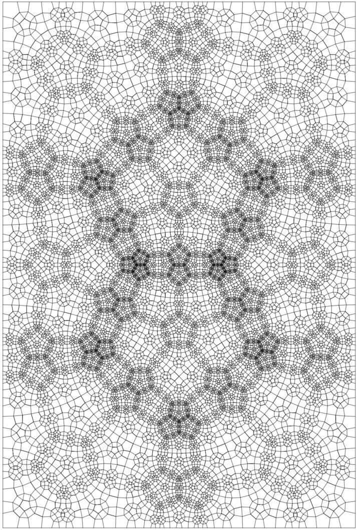

Source: Cannon - Floyd - Parry ’01

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Theorem [E - Lee '21]

For $\alpha, \epsilon > 0$, there is a unimodular random planar graph $G = (V, E)$ of almost sure uniform growth R^α that further almost surely satisfies

$$R_{\mathrm{eff}}\left(B(v,R), V \setminus B(v,10R)\right) \lesssim_\epsilon 1/R^{1-\epsilon}.$$

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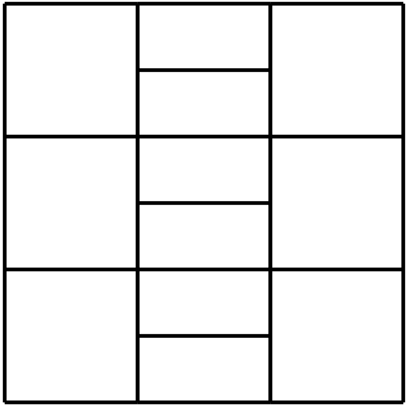
For all $\alpha > 2$, there is a *transient* planar graph of uniform growth R^α.

The Construction(s)

Tilings of $[0, 1]^2$



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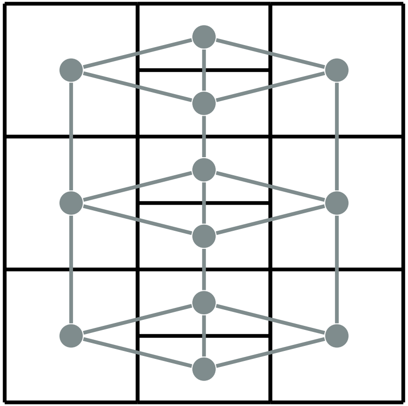
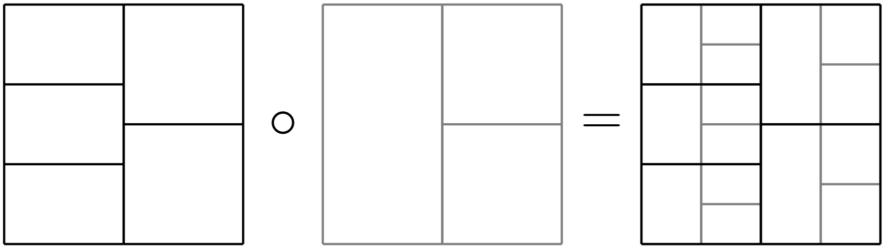


Tiling Product



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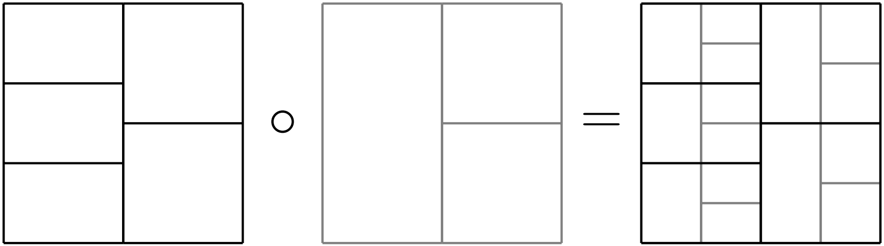
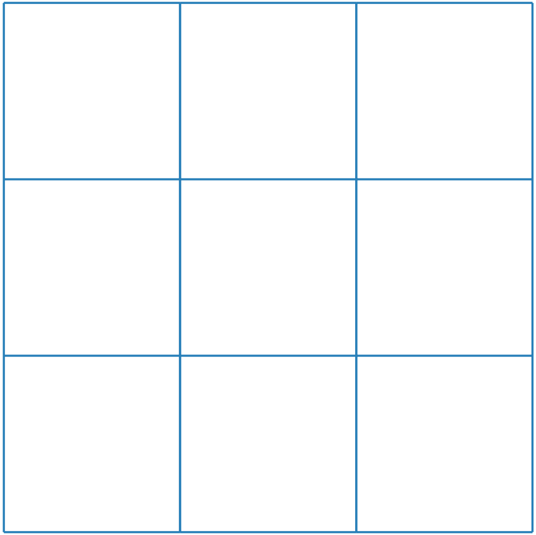


- Obesrvation: $A \circ (B \circ C) = (A \circ B) \circ C$.

$\{\mathbf{G}^n : n \geq 1\}$

$\mathbf{G} =$



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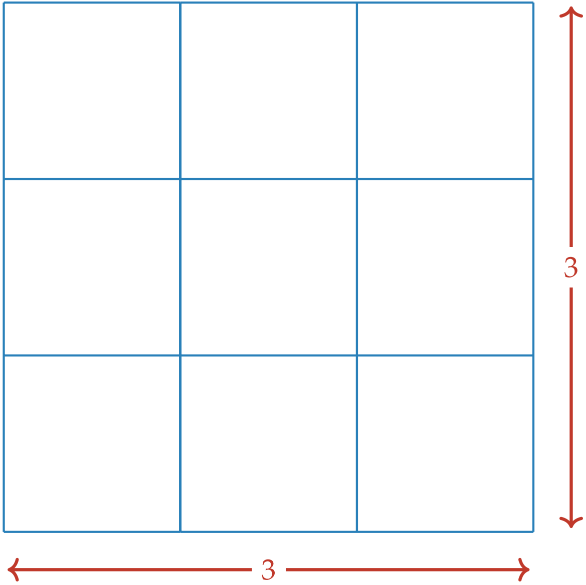

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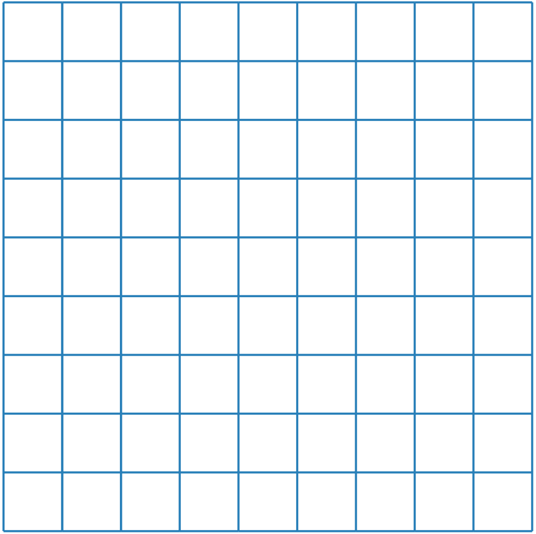

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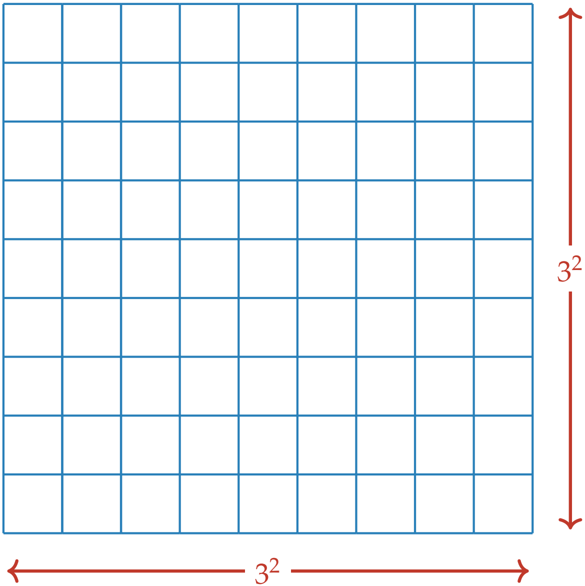

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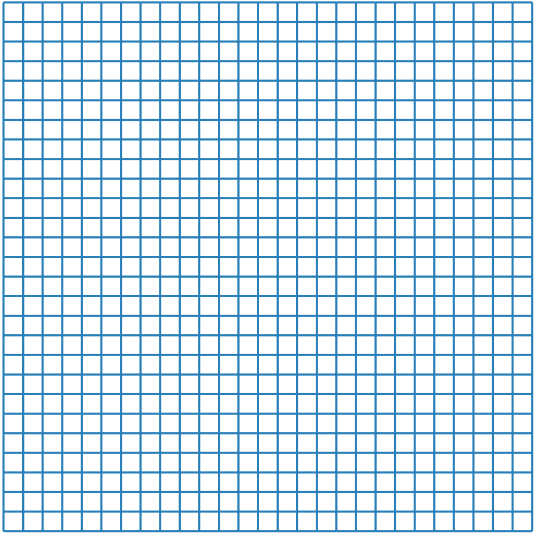

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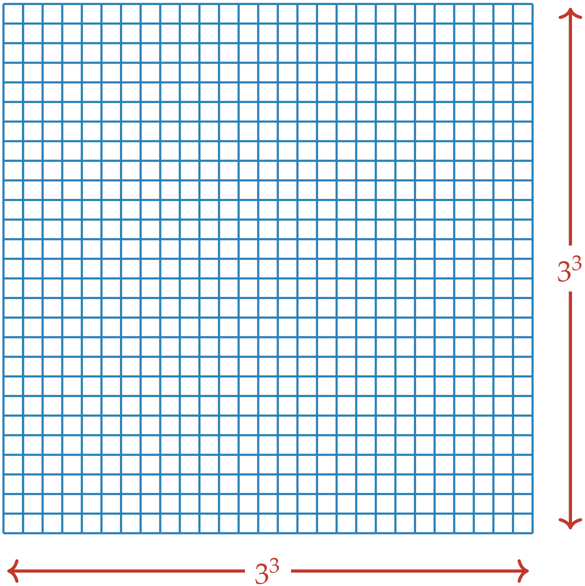

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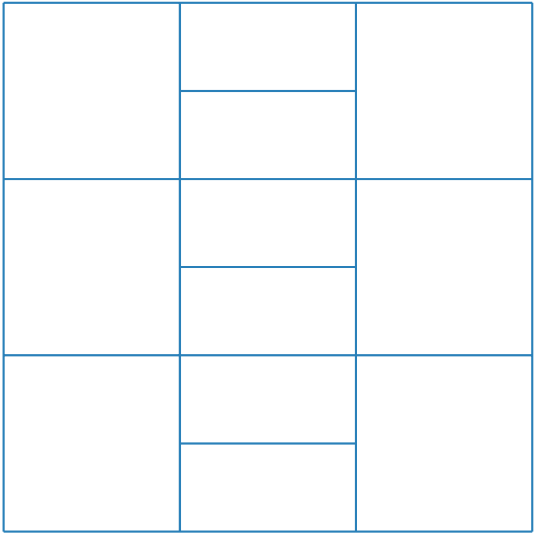
$\mathbf{H} =$ 

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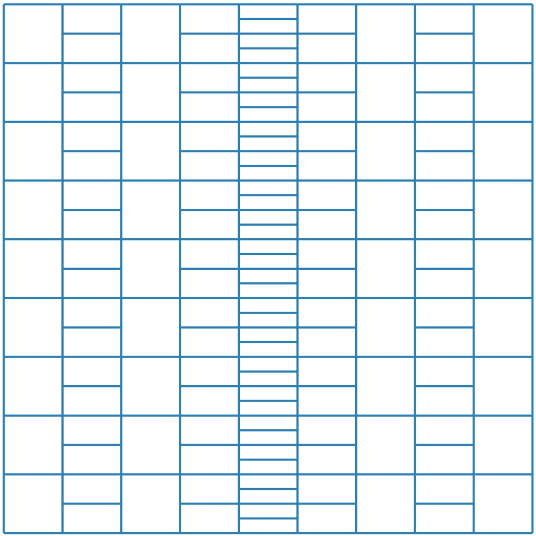
$\mathbf{H}^2 =$ 

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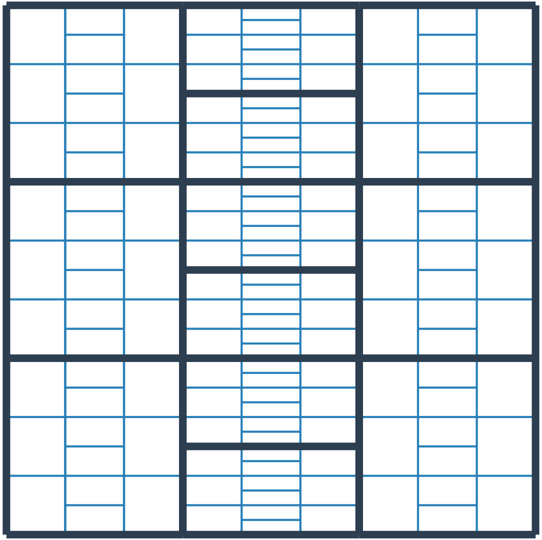

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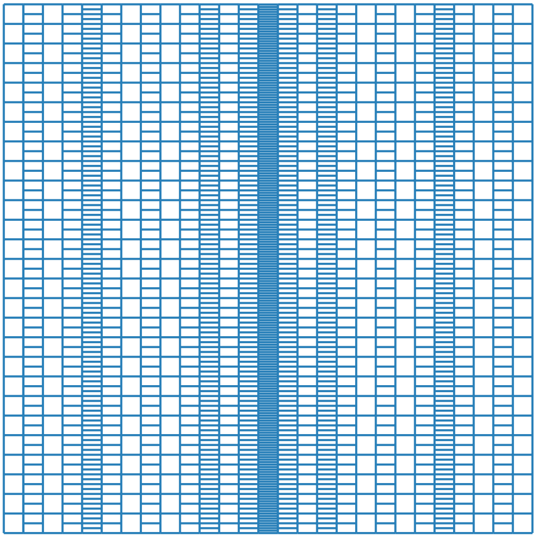
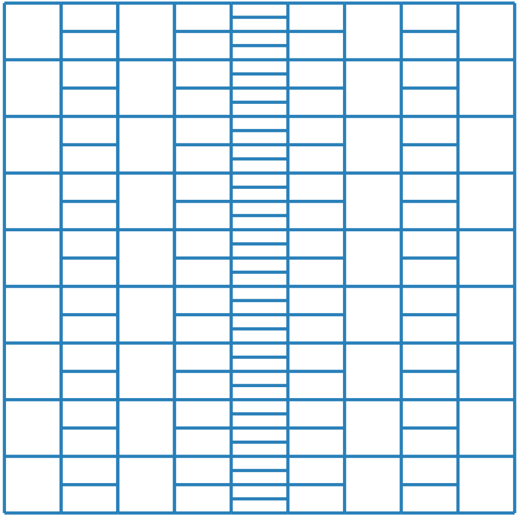


Lemma: $|B_{\mathbf{H}^n}(v, R)| \asymp R^{\log_3 12}$



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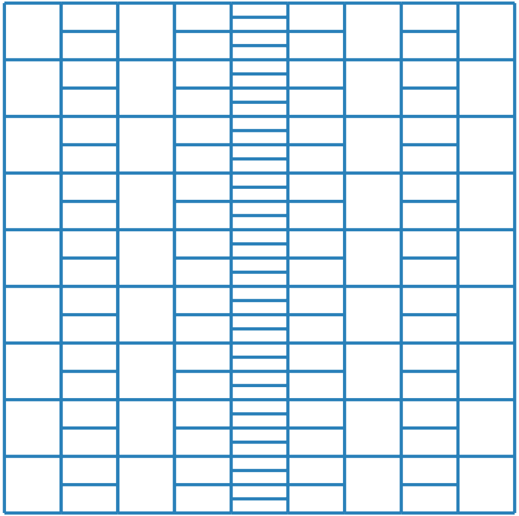

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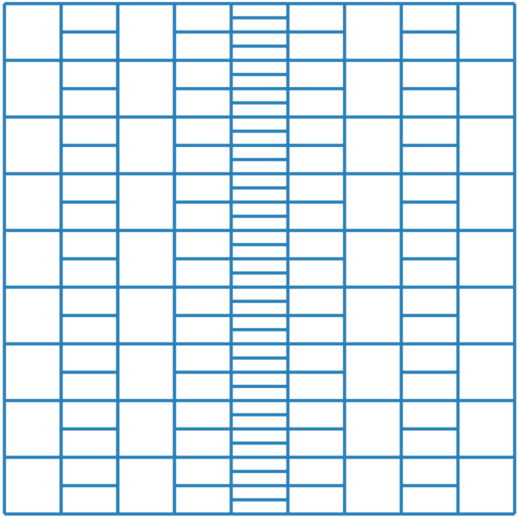

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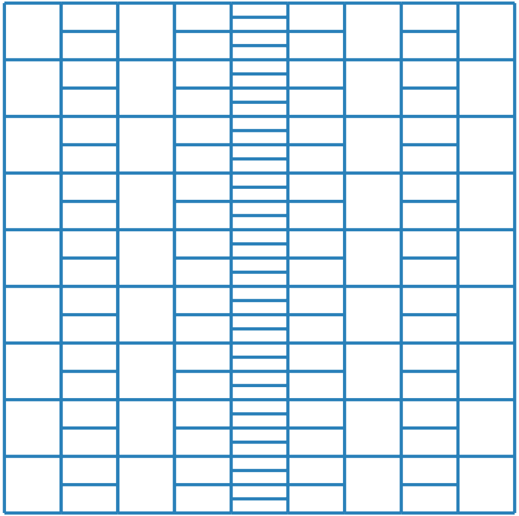

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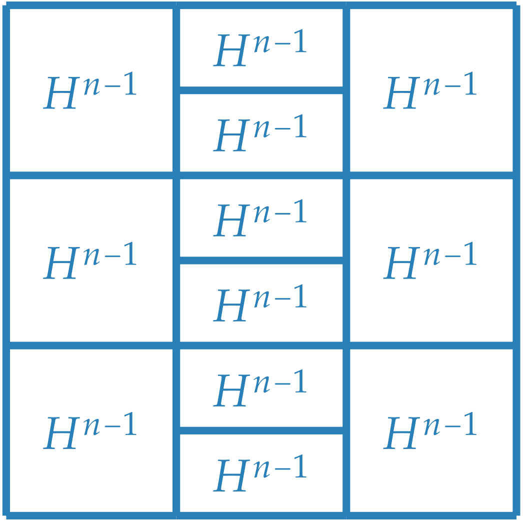

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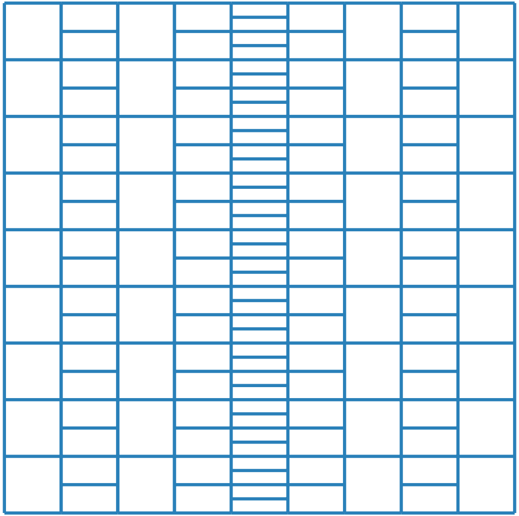

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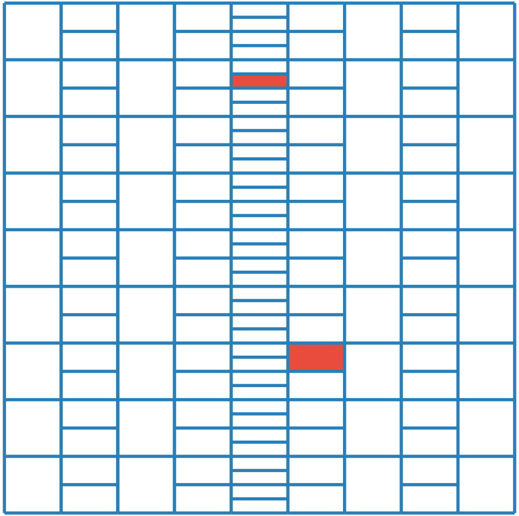

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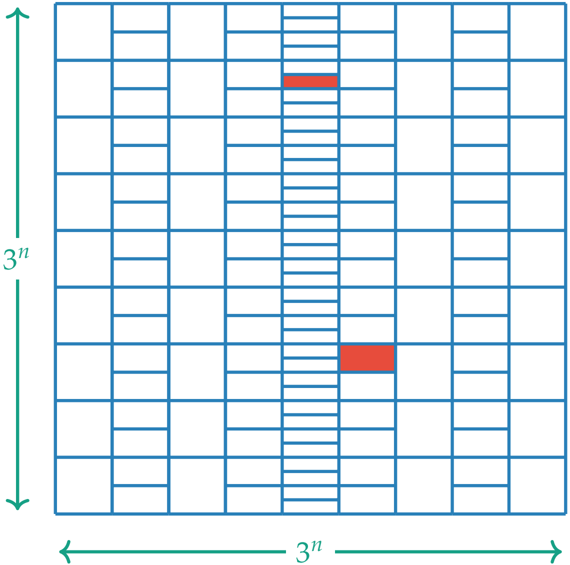

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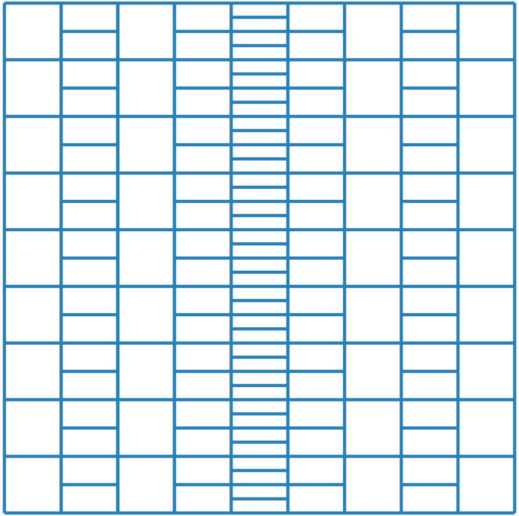

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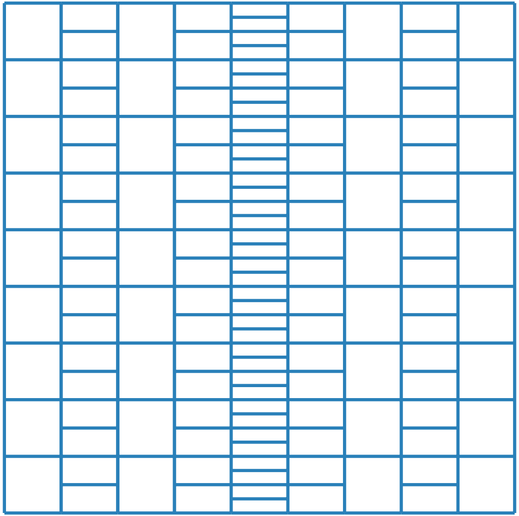

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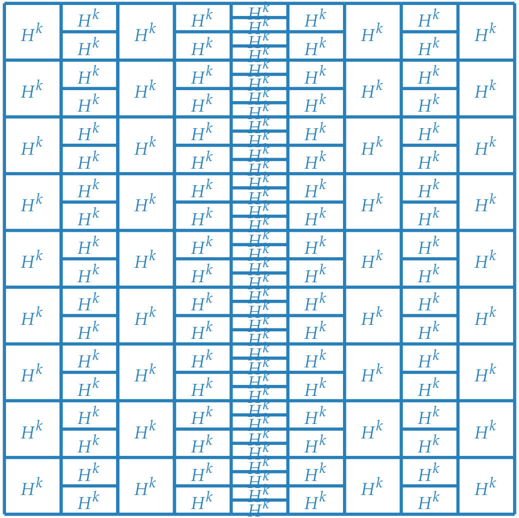

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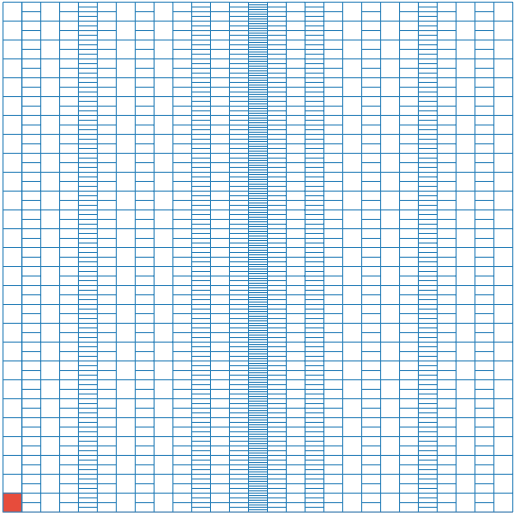

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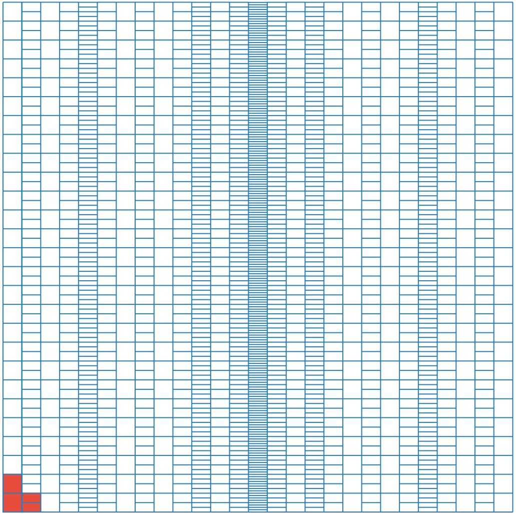

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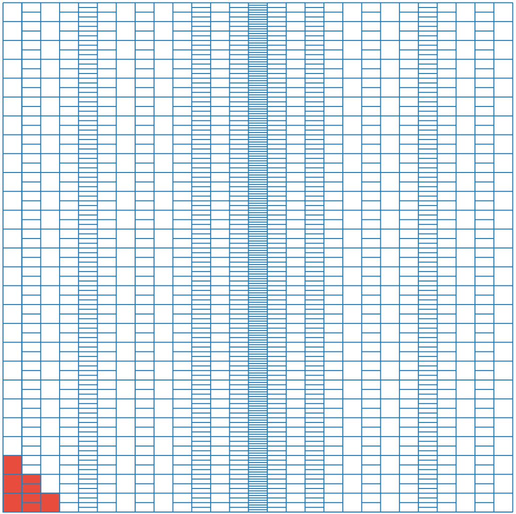

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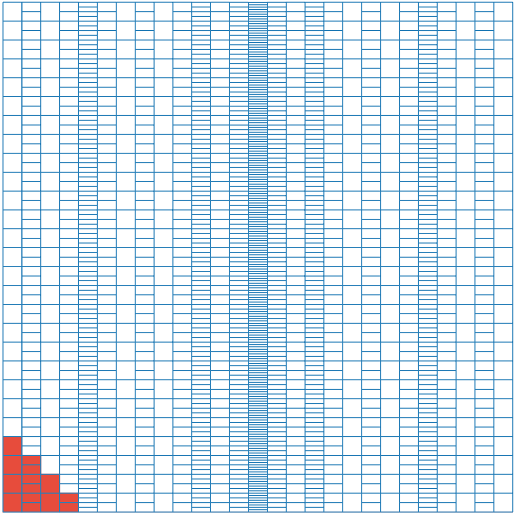

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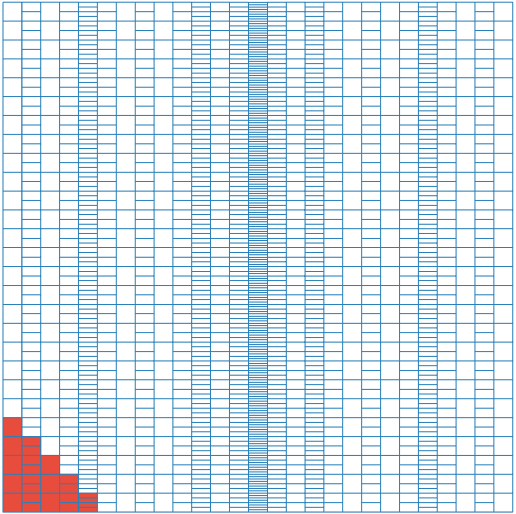

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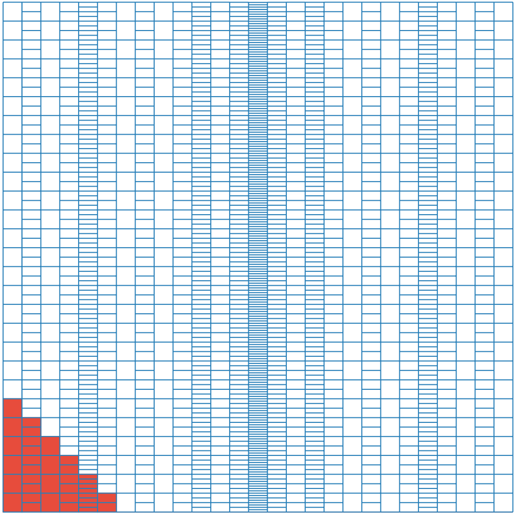

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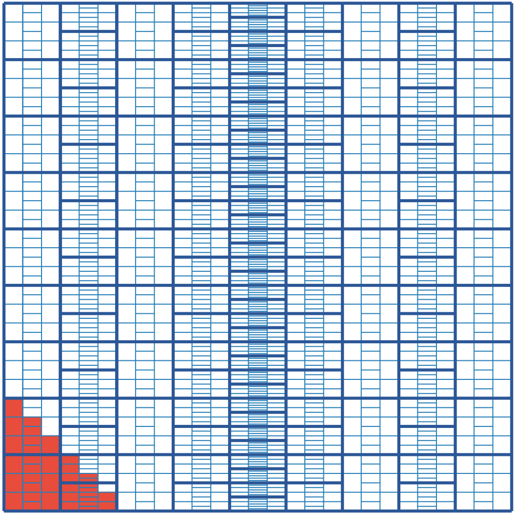

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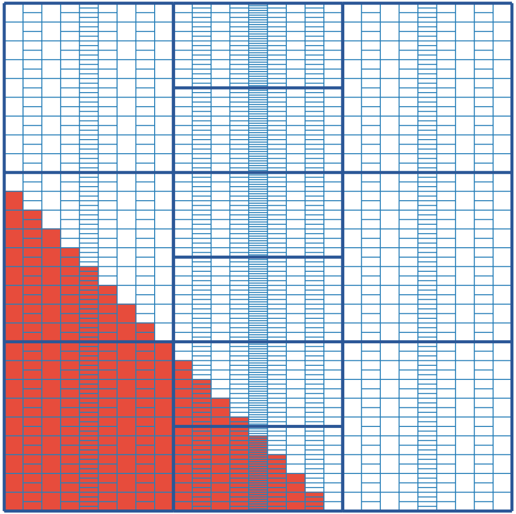

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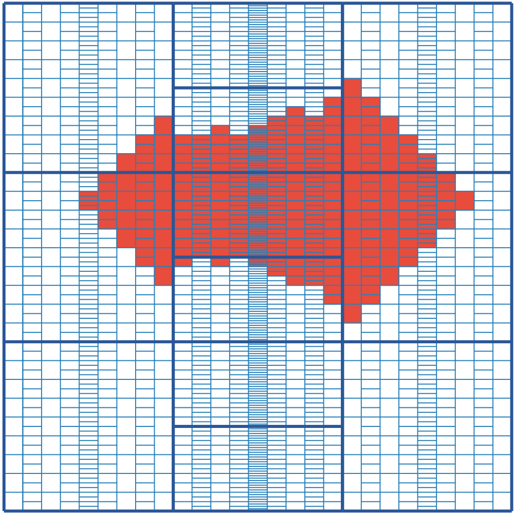

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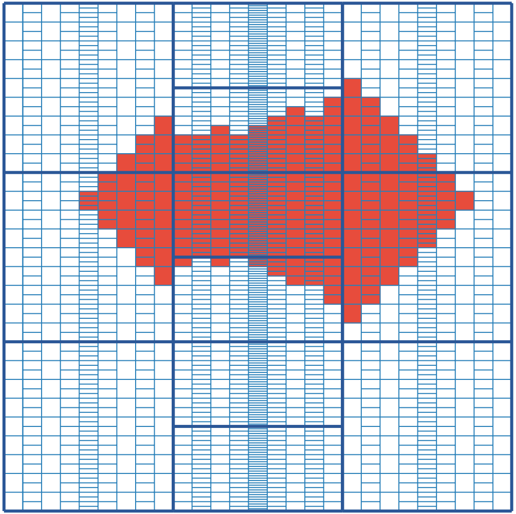

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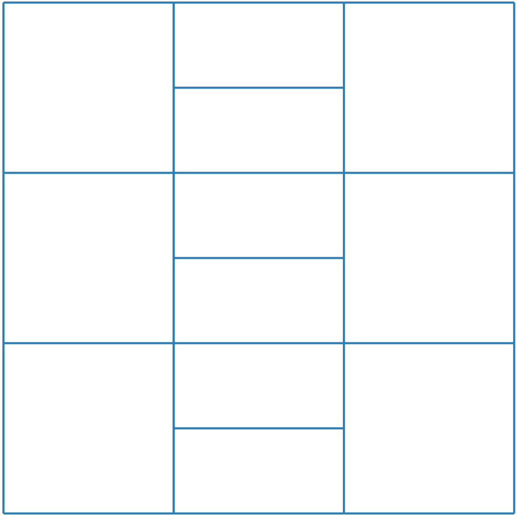

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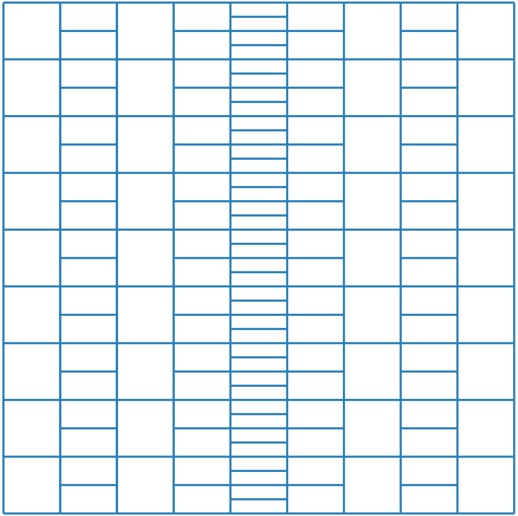

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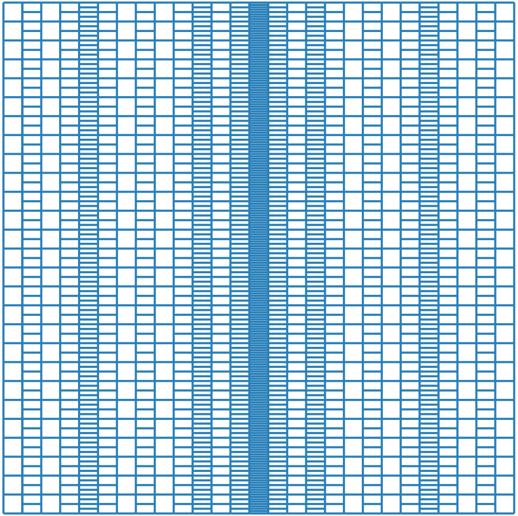

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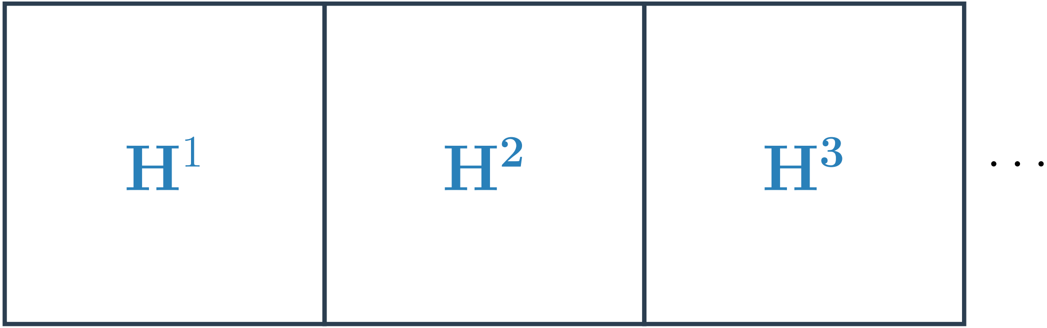

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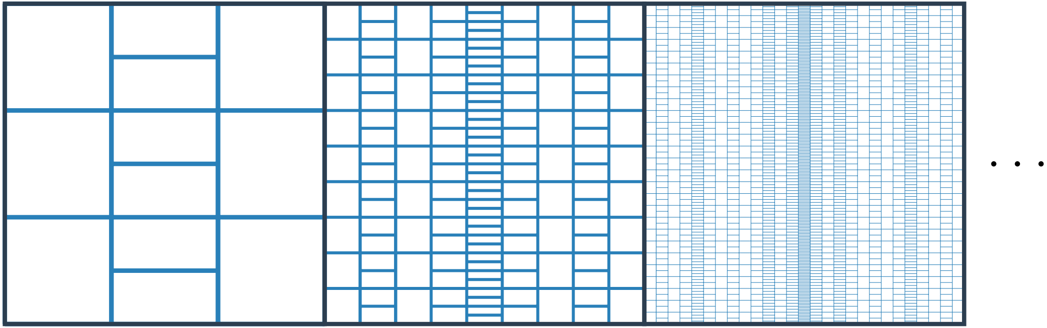

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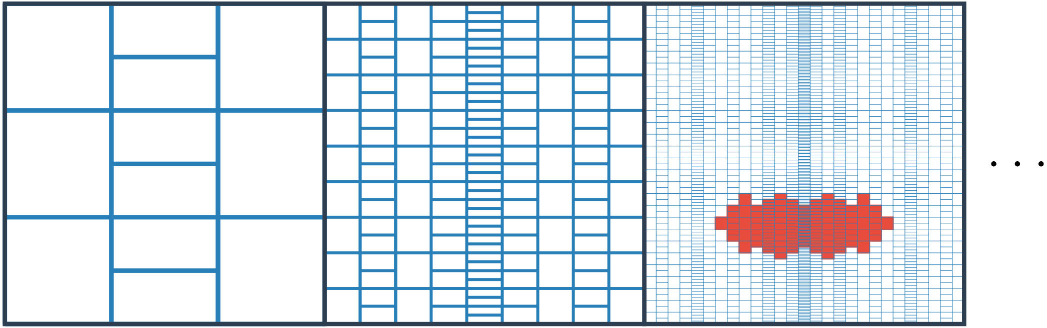

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\begin{aligned}
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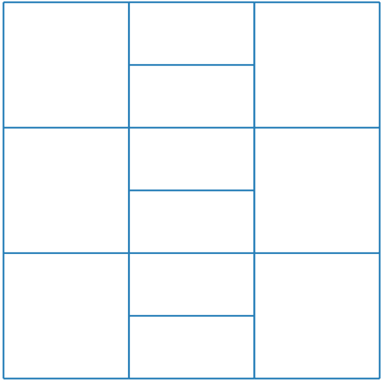
Transience of T_∞



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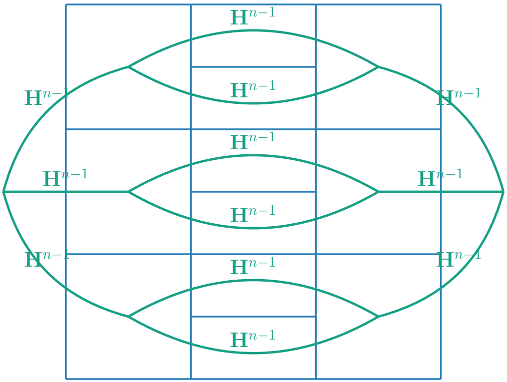

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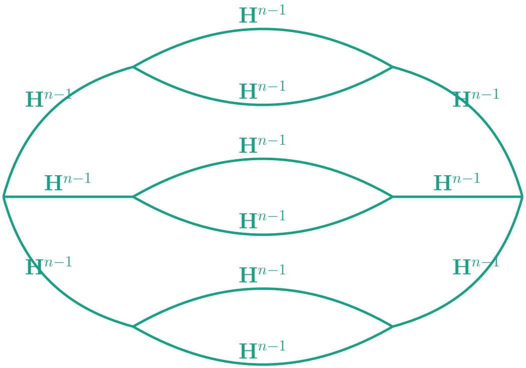

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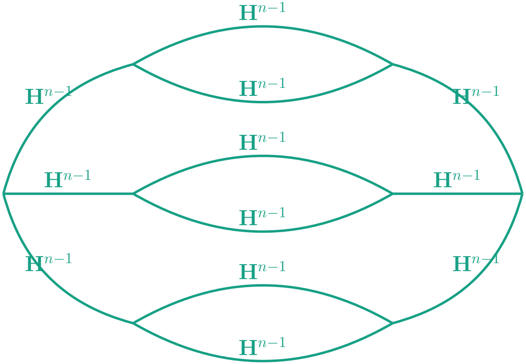

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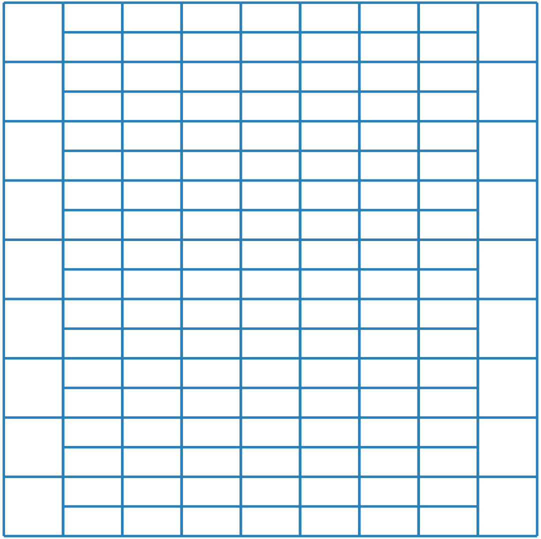

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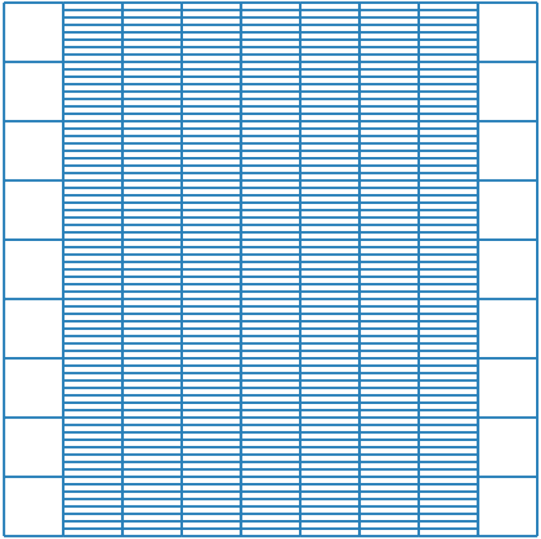

The Speed of the Random Walk

Theorem [Lee '17]

Suppose (G, ρ) is a unimodular random planar graph and G almost surely has uniform polynomial growth of degree d. Then:

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Theorem [E - Lee '21]

For $d \geq 2$ and $\epsilon > 0$, there is a unimodular random planar graph (G, ρ) that almost surely has uniform polynomial growth of degree d and

$$\mathbb{E}\left[d_G(X_0, X_t) \mid X_0 = \rho\right] \gtrsim_\epsilon t^{1/(\max(2,d-1)+\epsilon)}.$$

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- the complements of all balls are connected,
- the effective resistance across annuli is $1/R^{1-\epsilon}$,
- and the speed of the random walk is $t^{1/(\max(2,d-1)+\epsilon)}$.

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Questions?

Thank you!