Non-existence of annular separators in geometric graphs

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Abstract

Benjamini and Papasoglou (2011) showed that planar graphs with uniform polynomial volume growth admit 1-dimensional annular separators: The vertices at graph distance R from any vertex can be separated from those at distance $2R$ by removing at most $O(R)$ vertices. They asked whether geometric d-dimensional graphs with uniform polynomial volume growth similarly admit $(d-1)$-dimensional annular separators when $d > 2$. We show that this fails in a strong sense: For any $d \geqslant 3$ and every $s \geqslant 1$, there is a collection of interior-disjoint spheres in $\mathbb{R}^d$ whose tangency graph G has uniform polynomial growth, but such that all annular separators in G have cardinality at least R^s.

1 Introduction

The well-known Lipton-Tarjan separator theorem [LT79] asserts that any n-vertex planar graph has a balanced separator with $O(\sqrt{n})$ vertices. By the Koebe-Andreev-Thurston circle packing theorem, every planar graph can be realized as the tangency graph of interior-disjoint circles in the plane. One can define d-dimensional geometric graphs by analogy: Take a collection of "almost non-overlapping" bodies $\{S_v \subseteq \mathbb{R}^d : v \in V\}$, where each S_v is "almost round," and the associated geometric graph contains an edge $\{u, v\}$ if S_u and S_v "almost touch."

As a prototypical example, suppose we require that every point $x \in \mathbb{R}^d$ is contained in at most k of the bodies $\{S_v\}$, each S_v is a Euclidean ball, and two bodies are considered adjacent whenever $S_u \cap S_v \neq \emptyset$. These are precisely the intersection graphs of k-ply systems of balls, studied by Miller, Teng, Thurston, and Vavasis [MTTV97]. Those authors also provide a generalization of the Lipton-Tarjan separator theorem: For $k = O(1)$, such an intersection graph contains a balanced separator of size $O(n^{1-1/d})$.

Similarly, finite-element graphs associated to simplicial complexes with bounded aspect ratio can be viewed as subgraphs of geometric overlap graphs [MTTV98], and one again obtains balanced separators of size $O(n^{1-1/d})$. This covers a number of scenarios commonly arising in applications of the finite-element method; we refer to the discussion of well-shaped meshes in [ST07, §6.2].

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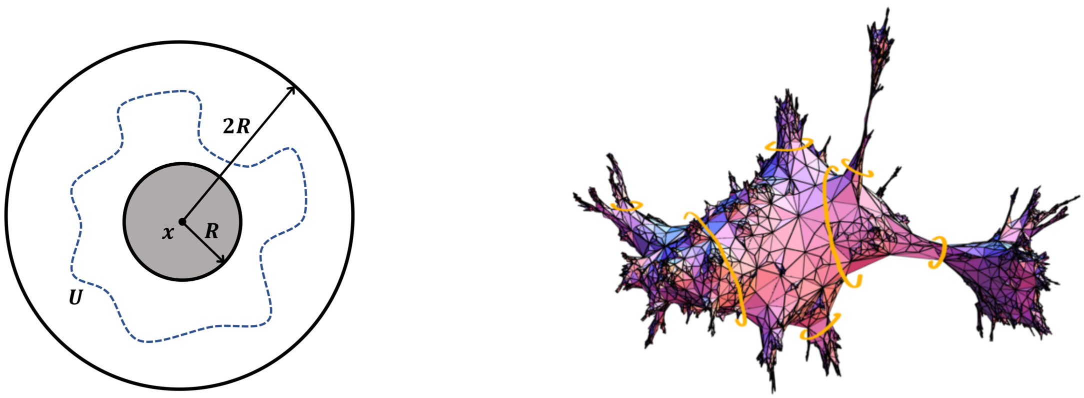
We will not be too concerned with the particular notion of geometric graph used since our construction satisfies all these commonly employed sets of assumptions. Indeed, it can be cast as the tangency graph of a sphere packing, where adjacent spheres have uniformly comparable radii. It can also be cast as the 1-skeleton of a d-dimensional simplicial complex whose simplices have uniformly bounded aspect ratio as studied. Such graphs were studied by, for instance, by Plotkin, Rao, and Smith [PRS94] in their work on shallow minors (see also the followup work [Ten98]).

Annular separators. Note that the preceding results deal with *global* separators that separate the entire graph into two roughly equal pieces. In many settings, especially those arising in physical simulation, it useful to consider *local* separators. Let $G = (V, E)$ be an undirected graph with path metric d_G, and define graph balls and graph spheres, respectively, by

$$B_G(x, R) := \{y \in V : d_G(x, y) \leqslant R\}$$
$$S_G(x, R) := \{y \in V : d_G(x, y) = R\}.$$

Suppose that for some $C > 1$, we we want to separate $S_G(x, R)$ from $S_G(x, CR)$ by removing a small set of nodes $U \subseteq B_G(x, CR) \setminus B_G(x, R)$. We refer to U as an *annular separator.* See Figure 1.



(a) Separating $S_G(x, R)$ from $S_G(x, 2R)$

(b) A random triangulation[1] of $\mathbb{S}^2$

Figure 1: Annular separators

It is easy to see that even if G is a planar graph, small annular separators don't necessary exist. But in many cases, one can find annular separators of size $O(R)$. For instance, this is true for most vertices at most scales in a uniformly random triangulation of the 2-dimensional sphere [Kri05] (a fact which extends experimentally to a variety of other models of random planar maps, e.g., those studied in [GHS20]). These models also have the properties that the cardinality of graph balls tends to grow asymptotically like $|B_G(x, R)| \sim R^k$ (as $R \to \infty$, up to lower-order fluctuations), where the exponent k depends on the model. For random triangulations, one has $k = 4$ [Ang03].

Benjamini and Papasoglou [BP11] give an explanation of this phenomenon as follows. Suppose that G is an infinite planar graph and we assume, additionally, that G *has uniform polynomial growth*

[1]Depiction of a random triangulation is due to Nicolas Curien.

of degree k: There exist numbers $C, k \geqslant 1$ such that

$$C^{-1}R^k \leqslant |B_G(x,R)| \leqslant CR^k, \quad \forall x \in V, R \geqslant 1.$$

Then for every $x \in V(G)$ and $R \geqslant 1$, there is a set $U \subseteq B_G(x,2R) \setminus B_G(x,R)$ whose removal disconnects $S_G(x,R)$ from $S_G(x,2R)$ in G, and such that $|U| \leqslant O(R)$. This applies equally well to finite graphs: Indeed, the authors actually show that if the graph metric restricted to $B_G(x,2R)$ has doubling constant λ, then one can find an annular separator of size at most $C_\lambda R$.

We remark that there are rich families of planar graphs with uniform polynomial growth arising in a variety of contexts; see [BS01, BK02, EL20]. Indeed, one can obtain planar graphs with uniform polynomial growth of degree k for all real degrees $k > 1$. Moreover, many models of random planar graphs have an almost sure asymptotic version of this property [DG20, GHS20].

The authors of [BP11] asked whether an analog of this phenomenon holds in higher dimensions. For instance, if G is a graph with uniform polynomial growth that can be geometrically represented in $\mathbb{R}^3$, does it hold that G has annular separators of size $O(R^2)$? We give examples showing that for $d \geqslant 3$, this phenomenon fails in a strong way.

Say that a graph G is *sphere-packed in* $\mathbb{R}^d$ if G is the tangency graph of a collection of interior-disjoint spheres in $\mathbb{R}^d$. Say that G is *M-uniformly sphere-packed in* $\mathbb{R}^d$ if the collection of spheres can be taken such that the radii of any two tangent spheres lies in the interval $[M^{-1}, M]$, and that G is *uniformly sphere-packed in* $\mathbb{R}^d$ if this holds for some $M < \infty$.

Theorem 1.1 (Arbitrarily large annular separators)**.** *For every $d \geqslant 3$ and $s \geqslant 1$, there is a number $c > 0$ and a graph G satisfying:*

1. *G has uniform polynomial growth of degree $O(s)$.*
2. *G is uniformly sphere-packed in $\mathbb{R}^d$.*
3. *For every $x \in V(G)$, there are at least cR^s vertex-disjoint paths from $S_G(x,R)$ to $S_G(x,R')$ for any $R' > R \geqslant 1$.*

Clearly if G has uniform polynomial growth of degree d, then one of the R-many spheres $S(x,R+1), S(x,R+2)\ldots, S(x,2R)$ must be an annular separator of size $O(R^{d-1})$. We show that the moment the growth degree exceeds d, there are graphs sphere-packed in $\mathbb{R}^d$ that don't have $(d-1)$-dimensional annular separators.

Theorem 1.2 (Nearly-dimensional growth rate)**.** *For every $d \geqslant 3$ and $\varepsilon > 0$, there are numbers $c, \delta > 0$ and a graph G satisfying:*

1. *G has uniform polynomial growth of degree at most $d + \varepsilon$.*
2. *G is uniformly sphere-packed in $\mathbb{R}^d$.*
3. *For every $x \in V(G)$, there are at least $cR^{(d-1)+\delta}$ vertex-disjoint paths from $S_G(x,R)$ to $S_G(x,R')$ for any $R' > R \geqslant 1$.*

Note that the two preceding theorems refer to infinite graphs. A version for families of finite graphs appears in Theorem 2.7.

Preliminaries. We will consider primarily connected, undirected graphs $G = (V, E)$, which we equip with the associated path metric d_G. We write $V(G)$ and $E(G)$, respectively, for the vertex and edge sets of G. If $U \subseteq V(G)$, we write $G[U]$ for the subgraph induced on U.

For $v \in V$, let $\deg_G(v)$ denote the degree of v in G. Let $\mathrm{diam}(G) := \sup_{x,y \in V} d_G(x, y)$ denote the diameter of G (which is only finite for G finite and connected), and for a subset $S \subseteq V$, denote $\mathrm{diam}_G(S) := \sup_{x,y \in S} d_G(x, y)$. For $v \in V$ and $r \geqslant 0$, we use $B_G(v, r) = \{u \in V : d_G(u, v) \leqslant r\}$ to denote the closed ball in G. For subsets $S, T \subseteq V$, we write $d_G(S, T) := \inf\{d_G(s, t) : s \in S, t \in T\}$.

For two expressions A and B, we use the notation $A \lesssim B$ to denote that $A \leqslant CB$ for some *universal* constant C. The notation $A \lesssim_{\alpha,\beta,\cdots} B$ denotes that $A \leqslant C(\alpha, \beta, \cdots)B$ where $C(\alpha, \beta, \cdots)$ denotes a number depending only on the parameters α, β, etc. We write $A \asymp B$ for the conjunction $A \lesssim B \wedge B \lesssim A$.

2 Tilings of the unit cube

Fix the dimension $d \geqslant 2$. Our constructions are based on tilings of subsets of $\mathbb{R}^d$ by axis-parallel hyperrectangles, a generalization of the planar constructions in [EL20]. A d-dimensional *tile* is an axis-parallel closed hyperrectangle $A \subseteq \mathbb{R}^d$, i.e., a set of the form $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ for numbers satisfying $a_i < b_i$ for each $i = 1, 2, \ldots, d$.

We will encode such a tile as a $(d + 1)$-tuple $(p(A), \ell_1(A), \ell_2(A), \ldots, \ell_d(A))$, where $p(A) := (a_1, a_2, \ldots, a_d)$ and $\ell_i(A) := b_i - a_i$ is the length of the projection of A along the ith axis.

A *tiling* $\boldsymbol{T}$ is a finite collection of interior-disjoint tiles. Denote $[\![\boldsymbol{T}]\!] := \bigcup_{A \in \boldsymbol{T}} A$. If $R \subseteq \mathbb{R}^d$, we say that $\boldsymbol{T}$ *is a tiling of* R if $[\![\boldsymbol{T}]\!] = R$. We associate to a tiling its *tangency graph* $G(\boldsymbol{T})$ with vertex set $\boldsymbol{T}$ and with an edge between two tiles $A, B \in \boldsymbol{T}$ whenever $A \cap B$ has non-zero $(d - 1)$-dimensional volume.

Denote by $\mathcal{T}_d$ the set of all tilings of the unit d-dimensional cube $[0, 1]^d$. See Figure 2(a) for a tiling of $[0, 1]^3$ and Figure 2(b) for a representation of its tangency graph. For the remainder of the paper, we will consider only tilings $\boldsymbol{T}$ for which $G(\boldsymbol{T})$ is connected.

Definition 2.1 (Tiling product)**.** For $\boldsymbol{S}, \boldsymbol{T} \in \mathcal{T}_d$, define the product $\boldsymbol{S} \circ \boldsymbol{T} \in \mathcal{T}_d$ as the tiling formed by replacing every tile in $\boldsymbol{S}$ by an (appropriately scaled) copy of $\boldsymbol{T}$. More precisely: For every $A \in \boldsymbol{S}$ and $B \in \boldsymbol{T}$, there is a tile $R \in \boldsymbol{S} \circ \boldsymbol{T}$ with $\ell_i(R) := \ell_i(A)\ell_i(B)$, and

$$p_i(R) := p_i(A) + p_i(B)\ell_i(A),$$

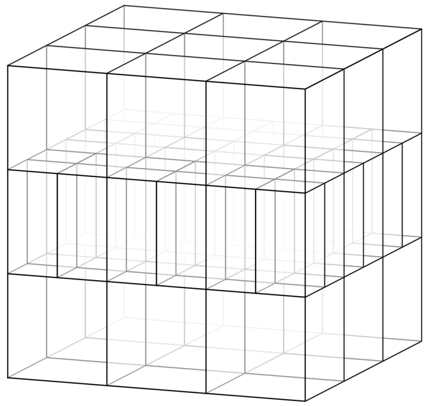
for each $i = 1, 2, \ldots, d$.

If $\boldsymbol{T} \in \mathcal{T}_d$ and $n \geqslant 0$, we use $\boldsymbol{T}^n := \boldsymbol{T} \circ \cdots \circ \boldsymbol{T}$ to denote the n-fold tile product of $\boldsymbol{T}$ with itself, where $\boldsymbol{T}^0 := \boldsymbol{I}_d$ and $\boldsymbol{I}_d := \{[0, 1]^d\} \in \mathcal{T}_d$ is the identity tiling. The following observation shows that this is well-defined.

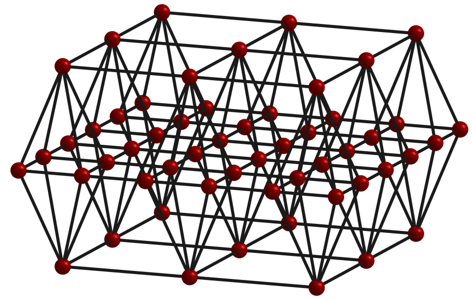
Observation 2.2. The tiling product is associative: $(\boldsymbol{S} \circ \boldsymbol{T}) \circ \boldsymbol{U} = \boldsymbol{S} \circ (\boldsymbol{T} \circ \boldsymbol{U})$ for all $\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{U} \in \mathcal{T}_d$. Moreover, if $\boldsymbol{I_d} \in \mathcal{T}_d$ consists of the single tile $[0, 1]^d$, then $\boldsymbol{T} \circ \boldsymbol{I_d} = \boldsymbol{I_d} \circ \boldsymbol{T}$ for all $\boldsymbol{T} \in \mathcal{T}_d$.

2.1 The construction

Given a sequence $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$ with $\gamma_i \in \mathbb{N}$, we define an associated tiling $\boldsymbol{T}_\gamma^{(d)} \in \mathcal{T}_d$ as follows: For $1 \leqslant i \leqslant b$, fill $[0, 1]^{d-1} \times [(i - 1)/b, i/b]$ with γ_i^{d-1} copies of $[0, 1/\gamma_i]^{d-1} \times [0, 1/b]$ formed into a



(a) $\boldsymbol{T}^{(3)}_{\langle 3,6,3\rangle}$



(b) The tangency graph

Figure 2: A tiling and its tangency graph

$\underbrace{\gamma_i \times \cdots \times \gamma_i}_{d-1 \text{ times}}$ grid. For example, see $\boldsymbol{T}^{(d)}_{\langle 3,6,3\rangle}$ in Figure 2(a). The following observation will be useful.

Observation 2.3. For $\gamma = \langle \gamma_1, \ldots, \gamma_a\rangle$, $\gamma' = \left\langle \gamma'_1, \ldots, \gamma'_b\right\rangle$, we have $\boldsymbol{T}^{(d)}_{\gamma} \circ \boldsymbol{T}^{(d)}_{\gamma'} = \boldsymbol{T}^{(d)}_{\gamma\otimes\gamma'}$ where

$$\gamma \otimes \gamma' = \left\langle \gamma_1\gamma'_1, \ldots, \gamma_1\gamma'_b, \ldots, \gamma_a\gamma'_1, \ldots, \gamma_a\gamma'_b\right\rangle.$$

In particular, for $n \geqslant 1$ it holds that $\left(\boldsymbol{T}^{(d)}_{\gamma}\right)^n = \boldsymbol{T}^{(d)}_{\gamma^{\otimes n}}$, where $\gamma^{\otimes n} := \gamma \otimes \gamma \otimes \cdots \otimes \gamma$ is the n-fold tensor product of γ with itself. See Figure 3(a) for a representation of $\boldsymbol{T}^{(3)}_{\langle 3,6,3\rangle^{\otimes 2}}$. We will use these two representations interchangeably throughout the paper.

We use the notations $|\gamma| := b$ and

$$|\gamma|^{(d)} := |\boldsymbol{T}^{(d)}_{\gamma}| = \gamma_1^{d-1} + \cdots + \gamma_b^{d-1}.$$

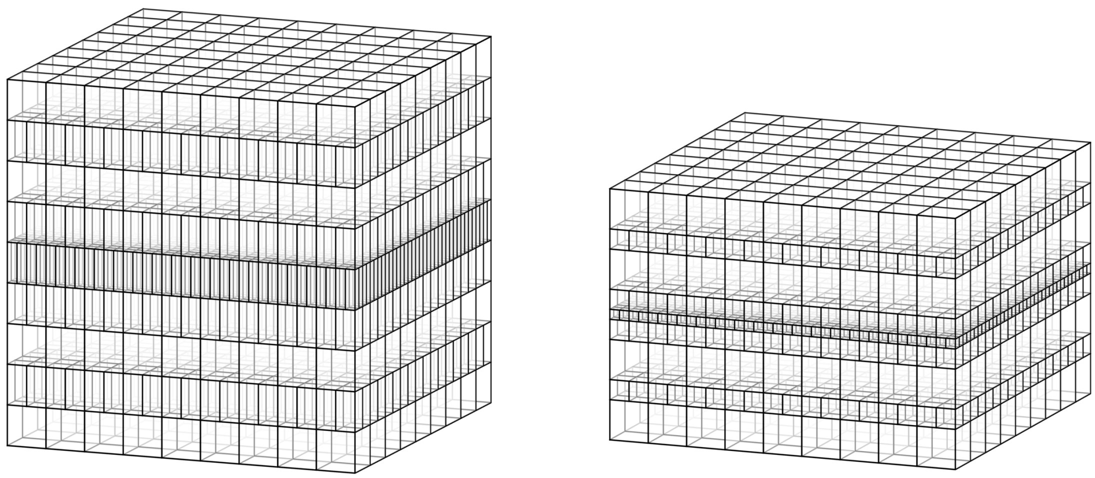
Note that $|\gamma \otimes \gamma'|^{(d)} = |\gamma|^{(d)}|\gamma'|^{(d)}$, hence

$$\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}} = |\gamma^{(d)}|^n. \tag{2.1}$$

Further, denote

$$\mathsf{k}^{(d)}(\gamma) := \frac{\log\left(|\gamma|^{(d)}\right)}{\log|\gamma|}.$$

We now state three key lemmas proved that are proved in subsequent sections, and then use them to prove our main theorems. The first establishes uniform polynomial volume growth for the graphs $\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}}$. It is proved in Section 3.



(a) $\boldsymbol{T}^{(3)}_{\langle 3,6,3\rangle} \circ \boldsymbol{T}^{(3)}_{\langle 3,6,3\rangle} = \boldsymbol{T}^{(3)}_{\langle 3,6,3\rangle^{\otimes 2}}$ (b) $\tilde{\boldsymbol{T}}^{(3)}_{\langle 3,6,3\rangle^{\otimes 2}}$

Figure 3: A tile product and its cube packing

Lemma 2.4 (Volume growth). *Consider $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$ with $\min(\gamma) = \gamma_1 = \gamma_b = b$. Then for all $d \geqslant 2$, there is a number $C = C(d, \gamma)$ such that the family of graphs $\mathcal{F} = \left\{ G\left(\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}}\right) : n \geqslant 0 \right\}$ has uniform polynomial growth of degree $k = \mathsf{k}^{(d)}(\gamma)$ in the sense that*

$$C^{-1} R^k \leqslant |B_G(x, R)| \leqslant C R^k, \qquad \forall G \in \mathcal{F}, x \in V(G), 1 \leqslant R \leqslant \operatorname{diam}(G).$$

The second lemma, proved in Section 4 establishes a lower bound on the size of annular separators.

Lemma 2.5 (Separator size). *For every $d \geqslant 2$ and $b \geqslant 1$, there is a number $c = c(d, b)$ such that for any $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$ with $\min(\gamma) = b$, the following holds. Denote $k := \mathsf{k}^{(d-1)}(\gamma)$. For any $n \geqslant 1$, if $G = G(\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}})$, and $v \in V(G)$, then there are at least cR^k disjoint paths from $S_G(v, R)$ to $S_G(v, R')$, for any $1 \leqslant R < R' \leqslant \operatorname{diam}(G)/(3(d+1))$.*

If G is an undirected graph, let us write $[G]_m$ for the *m-subdivision of G*, where each edge of G is replaced by a path of length m. We will sometimes consider $V(G) \subseteq V([G]_m)$ via the obvious identification. The next lemma is proved in Section 5.

Lemma 2.6. *Consider a sequence $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$ with $\gamma_1 = \gamma_b$, and such that*

$$\max\left\{ \frac{\gamma_{i+1}}{\gamma_i}, \frac{\gamma_i}{\gamma_{i+1}} \right\} \in \mathbb{N}, \quad \forall 1 \leqslant i < b. \tag{2.2}$$

Then for every $d \geqslant 2$, there are numbers $m = m(d, \gamma)$ and $M = M(d, \gamma)$ such that for every $n \geqslant 0$: If $G = G(\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}})$, then $[G]_m$ is M-uniformly sphere-packed in $\mathbb{R}^d$.

With these results in our hand, let us first prove a finitary version of our main theorems. The corresponding infinite version appears in Section 2.2. Define

$$s(d,k) := d - 1 + (k-d)\left(1 - \frac{1}{d-1}\right).$$

This represents our basic tradeoff: One can construct d-dimensional geometric graphs with uniform polynomial growth of degree $k+\varepsilon$ and such that every annular separator has size $\Omega(R^{s(d,k)})$.

Theorem 2.7 (Finite graph families)**.** *For every $d \geqslant 3$, $k \geqslant d$, and $\varepsilon > 0$, there is a family $\mathcal{F}$ of finite graphs satisfying:*

1. *For some $\tilde{k} \leqslant k+\varepsilon$ and every $G \in \mathcal{F}$,*

$$|B_G(x,R)| \asymp_{d,\varepsilon} R^{\tilde{k}}, \quad \forall x \in V(G), 1 \leqslant R \leqslant \mathrm{diam}(G).$$

2. *Each G is M-uniformly sphere-packed in $\mathbb{R}^d$ for some $M \lesssim_{d,\varepsilon} 1$.*
3. *There is a number $c \gtrsim_{d,\varepsilon} 1$ such that for every $G \in \mathcal{F}$ and $x \in V(G)$, there are at least $cR^{s(d,k)}$ vertex-disjoint paths from $S_G(x,R)$ to $S_G(x,R')$ for every $1 \leqslant R < R' \leqslant c\,\mathrm{diam}(G)$.*

In light of Lemma 2.4–Lemma 2.6, we can take $\mathcal{F} = \left\{[G(\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}})]_m : n \geqslant 0\right\}$ as long as we can find a sequence γ suited to the parameters. To this end, consider $d \geqslant 2$ and parameters $h, p, q \in \mathbb{N}$ such that $p \geqslant qd$. Define the sequence

$$\gamma^{(p,q,h)} := \langle b, \underbrace{tb, \dots, tb}_{b-2 \text{ copies}}, b\rangle,$$

where $b := h^{q(d-1)}$ and $t := h^{p-dq}$. Note that $t \geqslant 1$ since $p \geqslant qd$. Moreover, $\gamma^{(p,q,h)}$ satisfies (2.2) by construction.

Lemma 2.8. *Let $d \geqslant 2$ and $k := p/q$ be as above. Then for all $\varepsilon > 0$, there is some $h_0 \in \mathbb{N}$ so that for all $h \geqslant h_0$ the following statements hold:*

1. $\left|\mathsf{k}^{(d)}(\gamma^{(p,q,h)}) - k\right| < \varepsilon$.
2. $\left|\mathsf{k}^{(d-1)}(\gamma^{(p,q,h)}) - s(d,k)\right| < \varepsilon$.

Proof. It holds that

$$\begin{aligned}\mathsf{k}^{(d)}(\gamma^{(p,q,h)}) = \log_b(|\gamma^{(p,q,h)}|^{(d)}) &= \log_b\left(b^{d-1}(2+(b-2)t^{d-1})\right)\\ &= \log_b\left(b^{d-1}(2+(b-2)h^{(d-1)(p-dq)})\right)\\ &= \log_b\left(b^{d-1}(2+(b-2)b^{k-d})\right).\end{aligned}$$

Furthermore, we have

$$\lim_{h\to\infty}\log_b(|\gamma^{(p,q,h)}|^{(d)}) = \lim_{b\to\infty}\log_b(b^{d-1}(2+(b-2)b^{(k-d)})) = k.$$

Therefore for all $\varepsilon > 0$, for all sufficiently large values of h, it holds that $|\mathsf{k}^{(d)}(\gamma^{(p,q,h)}) - k| \leqslant \varepsilon$. Similarly, we have

$$\lim_{b\to\infty} \log_b(|\gamma^{(p,q,h)}|^{(d-1)}) = s(d,k),$$

hence we can choose h sufficiently large to as to satisfy the second condition as well. □

2.2 Construction of the infinite graphs

For this section alone, we consider tilings of the nonnegative orthant $[0,\infty)^d$.

Note that for any sequence $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$, one can view $G(\boldsymbol{T}_\gamma^{(d)})$ as the tangency graph of a packing of cubes by changing the height of cubes in layer i from $1/b$ to b/γ_i. See Figure 3(b) for an example. Let $\tilde{\boldsymbol{T}}_\gamma^{(d)}$ denote this rescaled tiling. By convention, we insist that one corner of the tiling still lies at the origin, implying that

$$[\![\tilde{\boldsymbol{T}}_\gamma^{(d)}]\!] = [0,1]^{d-1} \times [0, H(\gamma)],$$

where

$$H(\gamma) := b\left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_b}\right).$$

If we now assume that $\gamma_1 = \gamma_b = b$, then we have the chain of inclusions:

$$\tilde{\boldsymbol{T}}_{\gamma^{\otimes 0}}^{(d)} \subseteq \cdots \subseteq \tilde{\boldsymbol{T}}_{\gamma^{\otimes(n-1)}}^{(d)} \subseteq \tilde{\boldsymbol{T}}_{\gamma^{\otimes n}}^{(d)} \subseteq \tilde{\boldsymbol{T}}_{\gamma^{\otimes(n+1)}}^{(d)} \subseteq \cdots$$

This gives rise, in a straightforward way, to the infinite tiling $\tilde{\boldsymbol{T}}_{\gamma^{\otimes\mathbb{N}}}^{(d)}$, with $[\![\tilde{\boldsymbol{T}}_{\gamma^{\otimes\mathbb{N}}}^{(d)}]\!] = [0,\infty)^d$. We define the infinite tangency graph $\hat{G}_\gamma^{(d)} := G(\boldsymbol{T}_{\gamma^{\otimes\mathbb{N}}}^{(d)})$.

Theorem 2.9. *If $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$ has $\gamma_1 = \gamma_b = b = \min(\gamma)$, then $\hat{G}_\gamma^{(d)}$ has uniform polynomial growth of degree $\mathsf{k}^{(d)}(\gamma)$, and for every $x \in V(G)$ and $R' > R \geqslant 1$, there are at least $cR^{\mathsf{k}^{(d-1)}(\gamma)}$ vertex-disjoint paths from $S_G(x,R)$ to $S_G(x,R')$, where $c \gtrsim_{\gamma,d} 1$.*

Proof. For $n \geqslant 1$, denote $G_n := G(\boldsymbol{T}_{\gamma^{\otimes n}}^{(d)})$ and $\hat{G} := \hat{G}_\gamma^{(d)}$. We can think of G_n as an induced subgraph of $\hat{G}$ in the obvious way. Consider a vertex $v \in V(\hat{G})$ and radii $R' > R \geqslant 1$. Then there is some $n \geqslant 1$ such that $B_{\hat{G}}(v, 2R') \subseteq V(G_n)$.

In this case, the it holds that $d_{\hat{G}}(x,y) = d_{G_n}(x,y)$ for all $x, y \in B_{\hat{G}}(v,R')$, since any path originating in $B_{\hat{G}}(v,R')$ and leaving $V(G_n)$ must have length at least $2R'$. Therefore $B_{\hat{G}}(v,R') = B_{G_n}(v,R')$, and the uniform polynomial growth and seperator size assertions then follow immediately from Lemma 2.4 and Lemma 2.5. □

The following theorem is proved in Section 5.

Theorem 2.10. *If $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$ satisfies $\gamma_1 = \gamma_b$ and (2.2), then there is some number $m = m(d,\gamma)$ such that $[\hat{G}_\gamma^{(d)}]_m$ is uniformly sphere-packed in $\mathbb{R}^d$.*

3 Volume growth analysis

Our goal is now to prove Lemma 2.4. The next section provides a few key lemmas about the size of balls in products of tilings, which are mostly straightforward generalizations of the bounds in [EL20] (for the case $d = 2$). With these in hand, we prove Lemma 2.4 in Section 3.2.

3.1 Volume growth in tile products

The next lemma is straightforward.

Lemma 3.1. *Consider $\boldsymbol{S}, \boldsymbol{T} \in \mathcal{T}_d$ and $G = G(\boldsymbol{S} \circ \boldsymbol{T})$. If $X \in \boldsymbol{S} \circ \boldsymbol{T}$, then $|B_G(X, \mathrm{diam}(G(\boldsymbol{T})))| \geqslant |\boldsymbol{T}|$.*

Let $\{e_1, \ldots, e_d\}$ denote the standard basis of $\mathbb{R}^d$. If $\boldsymbol{T} \in \mathcal{T}_d$, we write $E_i(\boldsymbol{T}) \subseteq E(G(\boldsymbol{T}))$ for the set of edges in the ith direction, i.e., those edges $\{A, B\} \in E(G(\boldsymbol{T}))$ where $A \cap B$ is orthogonal to e_i. Thus we have a partition $E(G(\boldsymbol{T})) = E_1(\boldsymbol{T}) \cup \cdots \cup E_d(\boldsymbol{T})$.

For $A \in \boldsymbol{T}$ and $1 \leqslant i \leqslant d$, denote

$$N_{\boldsymbol{T}}(A, i) = \{A' \in \boldsymbol{T} : \{A, A'\} \in E_i(\boldsymbol{T})\},$$

and $N_{\boldsymbol{T}}(A) := N_{\boldsymbol{T}}(A, 1) \cup \cdots \cup N_{\boldsymbol{T}}(A, d)$. Moreover, we define

$$\begin{aligned}
\alpha_{\boldsymbol{T}}(A, i) &:= \max\left\{\frac{\ell_j(A)}{\ell_j(B)} : B \in N_{\boldsymbol{T}}(A, i), 1 \leqslant j \leqslant d\right\} \\
\alpha_{\boldsymbol{T}}(i) &:= \max\left\{\alpha_{\boldsymbol{T}}(A, i) : A \in \boldsymbol{T}\right\} \\
\alpha_{\boldsymbol{T}} &:= \max\left\{\alpha_{\boldsymbol{T}}(i) : 1 \leqslant i \leqslant d\right\} \\
L_{\boldsymbol{T}} &:= \max\left\{\ell_i(A) : A \in \boldsymbol{T}, 1 \leqslant i \leqslant d\right\}.
\end{aligned}$$

We will take $\alpha_{\boldsymbol{T}} := 1$ if $\boldsymbol{T}$ contains a single tile. It is now straightforward to check that $\alpha_{\boldsymbol{T}}$ bounds the degrees in $G(\boldsymbol{T})$.

Lemma 3.2. *For a tiling $\boldsymbol{T} \in \mathcal{T}_d$ and $A \in \boldsymbol{T}$, it holds that*

$$\deg_{G(\boldsymbol{T})}(A) \leqslant 3^d \cdot \alpha_{\boldsymbol{T}}^{2d}.$$

Proof. Denote

$$\tilde{A} := \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : p_i(A) - \alpha_{\boldsymbol{T}} \ell_i(A) \leqslant x_i \leqslant p_i(A) + (1 + \alpha_{\boldsymbol{T}}) \ell_i(A)\right\},$$

where $p_i(A)$ denotes the ith coordinate of $p(A)$. Clearly

$$\mathrm{vol}_d(\tilde{A}) = \prod_{i=1}^{d} ((1 + 2\alpha_{\boldsymbol{T}}) \ell_i(A)) = (1 + 2\alpha_{\boldsymbol{T}})^d \, \mathrm{vol}_d(A).$$

Furthermore, by the definition of $\alpha_{\boldsymbol{T}}$, it holds that $A' \subseteq \tilde{A}$ for $A' \in N_{\boldsymbol{T}}(A)$. And for any such A', it holds that $\mathrm{vol}_d(A') \geqslant \alpha_{\boldsymbol{T}}^{-d} \, \mathrm{vol}_d(A)$. Hence,

$$|N_{\boldsymbol{T}}(A)| \alpha_{\boldsymbol{T}}^{-d} \, \mathrm{vol}_d(A) \leqslant (1 + 2\alpha_{\boldsymbol{T}})^d \, \mathrm{vol}_d(A).$$

Using $\alpha_{\boldsymbol{T}} \geqslant 1$, this yields $\deg_{G(\boldsymbol{T})}(A) \leqslant 3^d \alpha_{\boldsymbol{T}}^{2d}$. $\square$

Lemma 3.3. *Consider $\boldsymbol{S}, \boldsymbol{T} \in \mathcal{T}_d$ and let $G = G(\boldsymbol{S} \circ \boldsymbol{T})$. Then for any $X \in \boldsymbol{S} \circ \boldsymbol{T}$, it holds that*

$$|B_G(X, 1/(\alpha_{\boldsymbol{S}}^{2d} L_{\boldsymbol{T}}))| \leqslant (3\alpha_{\boldsymbol{S}}^2)^{d^2}(d+1)|\boldsymbol{T}|. \tag{3.1}$$

Proof. For $Y \in \boldsymbol{S} \circ \boldsymbol{T}$, let $\hat{Y} \in \boldsymbol{S}$ denote the unique tile for which $Y \subseteq \hat{Y}$. Let $\Gamma_{\boldsymbol{S}}(\hat{X})$ denote the set of paths γ in $G(\boldsymbol{S})$ that originate from $\hat{X}$ and such that $|\gamma \cap E_i(\boldsymbol{S})| \leqslant 1$ for each $i = 1, \ldots, d$. In other words, the set of paths in $G(\boldsymbol{S})$ starting at $\hat{X}$ and containing at most one edge in every direction. Denote by $\tilde{N}_{\boldsymbol{S}}(\hat{X}) := \bigcup_{\gamma \in \Gamma_{\boldsymbol{S}}(\hat{X})} V(\gamma)$ the set of vertices reachable via such paths.

Note that because we allow one edge in every direction,

$$\hat{X} + [-\ell_1(\hat{X})/\alpha_{\boldsymbol{S}}^d, \ell_1(\hat{X})/\alpha_{\boldsymbol{S}}^d] \times \cdots \times [-\ell_d(\hat{X})/\alpha_{\boldsymbol{S}}^d, \ell_d(\hat{X})/\alpha_{\boldsymbol{S}}^d] \subseteq \left[\!\left[\tilde{N}_{\boldsymbol{S}}(\hat{X})\right]\!\right], \tag{3.2}$$

where '+' here is the Minkowski sum $R + S := \{r + s : r \in R, s \in S\}$.

We will now show that

$$\left[\!\left[B_G(X, 1/(\alpha_{\boldsymbol{S}}^{2d} L_{\boldsymbol{T}}))\right]\!\right] \subseteq \left[\!\left[\tilde{N}_{\boldsymbol{S}}(\hat{X})\right]\!\right]. \tag{3.3}$$

It will follow that

$$|B_G(X, 1/(\alpha_{\boldsymbol{S}}^{d+1} L_{\boldsymbol{T}}))| \leqslant |\boldsymbol{T}| \cdot |\tilde{N}_{\boldsymbol{S}}(\hat{X})| \leqslant |\boldsymbol{T}| \cdot (d+1) \left(\max_{A \in \boldsymbol{S}} \deg_{G(\boldsymbol{S})}(A)\right)^d,$$

and then (3.1) follows from Lemma 3.2.

To establish (3.3), consider any path $\langle X = X_0, X_1, X_2, \ldots, X_h \rangle$ in G with $\hat{X}_h \notin \tilde{N}_{\boldsymbol{S}}(\hat{X})$. Let $k \leqslant h$ be the smallest index for which $\hat{X}_k \notin \tilde{N}_{\boldsymbol{S}}(\hat{X})$. Then:

$$X_0, X_1, \ldots, X_{k-1} \subseteq \left[\!\left[\tilde{N}_{\boldsymbol{S}}(\hat{X})\right]\!\right] \tag{3.4}$$

$$X_{k-1} \cap \left(\partial \left[\!\left[\tilde{N}_{\boldsymbol{S}}(\hat{X})\right]\!\right] \cap (0,1)^d\right) \neq \emptyset. \tag{3.5}$$

Now (3.4) implies that for $j \leqslant k-1$, we have $\hat{X}_j \in \tilde{N}_{\boldsymbol{S}}(\hat{X})$, which implies $\ell_i(\hat{X}_j) \leqslant \alpha_{\boldsymbol{S}}^d \ell_i(\hat{X})$ since $\hat{X}_j$ can be reached from $\hat{X}$ by a path of length at most d in $G(\boldsymbol{S})$. It follows that

$$\ell_i(X_j) \leqslant L_{\boldsymbol{T}} \ell_i(\hat{X}_j) \leqslant L_{\boldsymbol{T}} \alpha_{\boldsymbol{S}}^d \ell_i(\hat{X}), \qquad j \leqslant k-1, 1 \leqslant i \leqslant d. \tag{3.6}$$

And (3.5) together with (3.2) shows that

$$\sum_{j=0}^{k-1} \ell_i(X_j) \geqslant \ell_i(\hat{X})/\alpha_{\boldsymbol{S}}^q, \qquad 1 \leqslant i \leqslant d. \tag{3.7}$$

Combining (3.6) and (3.7) now gives

$$h - 1 \geqslant k - 1 \geqslant \frac{1}{\alpha_{\boldsymbol{S}}^{2d} L_{\boldsymbol{T}}},$$

verifying (3.3) and completing the proof. □

3.2 Volume growth in iterated products

Our goal is now to prove Lemma 2.4. To this end, fix $d \geqslant 2$.

Observation 3.4. For $1 \leqslant i \leqslant d-1$, we have $\alpha_{T_\gamma^{(d)}}(i) = 1$. It further holds that

$$\alpha_{T_\gamma^{(d)}} = \alpha_{T_\gamma^{(d)}}(1) = \max\left\{\frac{\gamma_i}{\gamma_j} : 1 \leqslant i,j \leqslant b, |i-j| = 1\right\}.$$

Given Observation 2.3 and Observation 3.4, the next lemma is straightforward.

Lemma 3.5. *Consider $\gamma = \langle \gamma_1, \dots, \gamma_b\rangle$ and $\gamma' = \left\langle \gamma'_1, \dots, \gamma_{b'}\right\rangle$. If $\gamma'_1 = \gamma'_{b'}$, then*

$$\alpha_{T_\gamma^{(d)} \circ T_{\gamma'}^{(d)}} = \alpha_{T_{\gamma\otimes\gamma'}^{(d)}} \leqslant \max(\alpha_{T_\gamma^{(d)}}, \alpha_{T_{\gamma'}^{(d)}}).$$

Proof. Note that, by definition for $1 \leqslant j \leqslant b'$ and $0 \leqslant i \leqslant b-1$, we have $(\gamma\otimes\gamma')_{ib'+j} = \gamma_i\gamma'_j$. Hence,

$$\frac{(\gamma\otimes\gamma')_{ib'+j+1}}{(\gamma\otimes\gamma')_{ib'+j}} = \begin{cases} \frac{\gamma'_{j+1}}{\gamma'_j} & j < b' \\ \frac{\gamma_{i+1}}{\gamma_i}\frac{\gamma'_1}{\gamma'_{b'}} = \frac{\gamma_{i+1}}{\gamma_i} & j = b', \end{cases}$$

where we used $\gamma'_1 = \gamma'_{b'}$ in the second case. □

Corollary 3.6. *Let $\gamma = \langle \gamma_1, \dots, \gamma_b\rangle$. If $\gamma_1 = \gamma_b$, then for $n \geqslant 1$ we have*

$$\alpha_{T_{\gamma^{\otimes n}}^{(d)}} \leqslant \alpha_{T_\gamma^{(d)}}.$$

Lemma 3.7. *Consider $\gamma = \langle \gamma_1, \dots, \gamma_b\rangle$ with $\gamma_1 = \gamma_b = b$. For every $n \geqslant 0$, it holds that*

$$b^n - 1 \leqslant \operatorname{diam}\left(G(T_{\gamma^{\otimes n}}^{(d)})\right) \leqslant (d+1)b^n. \tag{3.8}$$

Proof. Let C denote the "bottom" layer of tiles in $T_{\gamma^{\otimes n}}^{(d)}$, i.e., those contained in $[0,1]^{d-1}\times[0,1/b^n]$. Take $G := G(T_{\gamma^{\otimes n}}^{(d)})$. For every $A \in T_{\gamma^{\otimes n}}^{(d)}$, it holds that $\ell_d(A) = b^{-n}$. In particular, this yields the lower bound in (3.8) since any path from the bottom to the top (in dimension d) requires b^n tiles.

This also implies that for $A \in T_{\gamma^{\otimes n}}^{(d)}$ we have $d_G(A, C) \leqslant b^n$. Furthermore, as $\gamma_1 = b$ by assumption, we have $(\gamma^{\otimes n})_1 = b^n$, and therefore by construction $G[C]$ is a $(d-1)$-dimensional $b^n \times b^n \times \cdots \times b^n$ grid. Hence we have $\operatorname{diam}_G(C) \leqslant (d-1)b^n$, and now the upper bound in (3.8) follows by the triangle inequality. □

Proof of Lemma 2.4. Denote $G := G(T_{\gamma^{\otimes n}}^{(d)})$ and fix $A \in T_{\gamma^{\otimes n}}^{(d)}$. For any $0 \leqslant j \leqslant n$, write $T_{\gamma^{\otimes n}}^{(d)} = T_{\gamma^{\otimes(n-j)}}^{(d)} \circ T_{\gamma^{\otimes j}}^{(d)}$ and let $\hat{A}$ denote the copy of $T_{\gamma^{\otimes j}}^{(d)}$ containing A. By Lemma 3.7, we have $\operatorname{diam}(\hat{A}) \leqslant (d+1)b^j$, and thus $B_G(A, (d+1)b^j) \supseteq \hat{A}$.

Employing Lemma 3.1 therefore gives

$$\left|B_G(A, (d+1)b^j)\right| \geqslant |\hat{A}| = \left|T_{\gamma^{\otimes j}}^{(d)}\right| = \left(|\gamma|^{(d)}\right)^j, \qquad \forall A \in T_{\gamma^{\otimes n}}^{(d)},\ j \in \{0,1,\dots,n\}.$$

The desired lower bound now follows using monotonicity of $|B_G(A,r)|$ with respect to r.

To prove the upper bound, first note that by Corollary 3.6 we have

$$\alpha_{\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}}} \leqslant \alpha_{\boldsymbol{T}^{(d)}_{\gamma}} \lesssim_{\gamma} 1.$$

Moreover, as $\min(\gamma) = b$, we have $L_{\boldsymbol{T}^{(d)}_{\gamma^{\otimes j}}} = b^{-j}$, hence applying Lemma 3.3 with $S = \boldsymbol{T}^{(d)}_{\gamma^{\otimes(n-j)}}$ and $T = \boldsymbol{T}^{(d)}_{\gamma^{\otimes j}}$ gives

$$\left|B_G(A, b^j/\xi)\right| \lesssim_{d,\gamma} |\boldsymbol{T}^{(d)}_{\gamma^{\otimes j}}| = \left(|\gamma|^{(d)}\right)^j, \qquad \forall A \in \boldsymbol{T}^{(d)}_{\gamma^{\otimes n}},\ j \in \{0,1,\dots,n\},$$

for some $\xi \lesssim_{\gamma,d} 1$, completing the proof. □

4 The size of annular separators

We now prove that the graphs $G(\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}})$ do not have small annular separators.

Definition 4.1 (Projection of tiles)**.** For $d \geqslant 2$ and a tile $A \subseteq \mathbb{R}^d$, we write $\Pi_{d-1}(A) \subseteq \mathbb{R}^{d-1}$ for the projection of A onto the last $d-1$ coordinates. Furthermore, for a tiling $\boldsymbol{T} \in \mathcal{T}_d$, we define

$$\Pi_{d-1}(\boldsymbol{T}) := \{\Pi_{d-1}(A) \mid A \in \boldsymbol{T}\},$$

and for a tile $B \in \Pi_{d-1}(\boldsymbol{T})$,

$$\Pi^{-1}_{d-1}(B;\boldsymbol{T}) := \{A \in \boldsymbol{T} : \Pi_{d-1}(A) = B\}.$$

Observation 4.2. For any sequence $\gamma = \langle \gamma_1, \dots, \gamma_b \rangle$, the following hold:

(a) $\Pi_{d-1}(\boldsymbol{T}^{(d)}_{\gamma}) = \boldsymbol{T}^{(d-1)}_{\gamma}$.

(b) For all $B \in \Pi_{d-1}(\boldsymbol{T}^{(d)}_{\gamma})$, the tangency graph $G(\Pi^{-1}_{d-1}(B;\boldsymbol{T}^{(d)}_{\gamma}))$ is a path.

(c) For all $B \neq B' \in \Pi_{d-1}(\boldsymbol{T}^{(d)}_{\gamma})$, the sets $\Pi^{-1}_{d-1}(B;\boldsymbol{T}^{(d)}_{\gamma})$ and $\Pi^{-1}_{d-1}(B';\boldsymbol{T}^{(d)}_{\gamma})$ are disjoint.

See Figure 4.

We will now use the family $\{\Pi^{-1}_{d-1}(B;\boldsymbol{T}^{(d)}_{\gamma}) : B \in \Pi_{d-1}(\boldsymbol{T}^{(d)}_{\gamma})\}$ of pairwise disjoint paths to locate paths across annuli in $G(\boldsymbol{T}^{(d)}_{\gamma^{\otimes n}})$. We first remark that diameter of each such path is long in G.

Lemma 4.3. *For every $B \in \Pi_{d-1}(\boldsymbol{T}^{(d)}_{\gamma})$,*

$$\operatorname{diam}_{G(\boldsymbol{T}^{(d)}_{\gamma})}(\Pi^{-1}_{d-1}(B;\boldsymbol{T}^{(d)}_{\gamma})) \geqslant \frac{1}{L_{\boldsymbol{T}^{(d)}_{\gamma}}} - 1,$$

Proof. Let $\Pi_1 : \mathbb{R}^d \to \mathbb{R}$ denote the projection onto the first coordinate. Then the Euclidean diameter of $\Pi_1(\Pi^{-1}_{d-1}(B;\boldsymbol{T}^{(d)}_{\gamma}))$ is precisely 1. On the other hand, for any path P in $G(\boldsymbol{T}^{(d)}_{\gamma})$, the Euclidean diameter of $\Pi_1(P)$ is at most $(\operatorname{len}(P)+1)L_{\boldsymbol{T}^{(d)}_{\gamma}}$, and the result follows. □

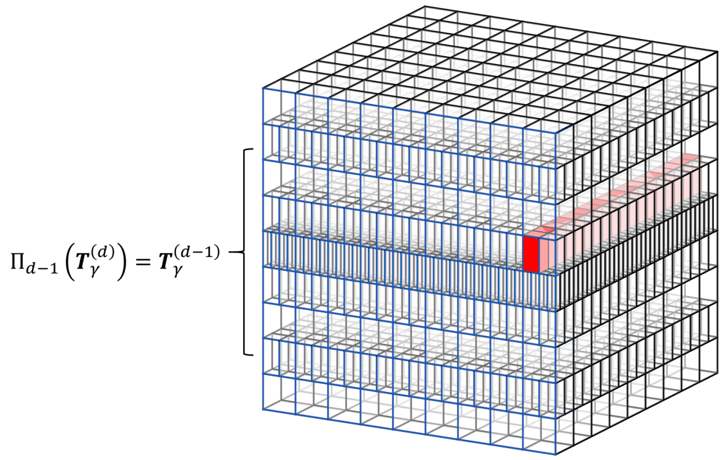


Figure 4: Projection to the last $d-1$ coordinates and $\Pi_{d-1}^{-1}(B; \boldsymbol{T}_\gamma^{(d)})$ for one $B \in \Pi_{d-1}(\boldsymbol{T}_\gamma^{(d)})$.

With this in hand, we can now exhibit many disjoint paths across annuli in G.

Proof of Lemma 2.5. Define $G := G(\boldsymbol{T}_{\gamma^{\otimes n}}^{(d)})$ and $h := \lfloor \log_b(R/(d+1)) \rfloor$. Let $A \in \boldsymbol{T}_{\gamma^{\otimes n}}^{(d)}$ be arbitrary. Write $\boldsymbol{T}_{\gamma^{\otimes n}}^{(d)} = \boldsymbol{T}_{\gamma^{\otimes(n-h)}}^{(d)} \circ \boldsymbol{T}_{\gamma^{\otimes h}}^{(d)}$, and let $\hat{A} \in \boldsymbol{T}_{\gamma^{\otimes n}}^{(d)}$ be the copy of $\boldsymbol{T}_{\gamma^{\otimes h}}^{(d)}$ that contains A. By Lemma 3.7, we have $\operatorname{diam}_G(\hat{A}) \leqslant (d+1)b^h \leqslant R$, where the latter inequality follows from our definition of h. Thus $\hat{A} \subseteq B_G(A, R)$.

But as $\hat{A}$ is a translation of $\boldsymbol{T}_{\gamma^{\otimes h}}^{(d)}$, by Observation 4.2(a), it holds that $\Pi_{d-1}(\boldsymbol{T}_{\gamma^{\otimes h}}^{(d)})$ is a translation of $\boldsymbol{T}_{\gamma^{\otimes h}}^{(d-1)}$. This yields

$$|\Pi_{d-1}\left(B_G(A,R)\right)| \geqslant |\Pi_{d-1}(\hat{A})| = |\boldsymbol{T}_{\gamma^{\otimes h}}^{(d-1)}| = \left(|\gamma|^{(d-1)}\right)^h.$$

Using Observation 4.2(b)–(c), the sets $\{\Pi_{d-1}^{-1}(B; \boldsymbol{T}_{\gamma^{\otimes n}}^{(d)}) : B \in \Pi_{d-1}(B_G(A,R))\}$ form a collection of vertex-disjoint paths in G, and Lemma 4.3 gives a lower bound on the diameter of every such path in G. It follows that for

$$R' < \frac{1}{2}\left(\frac{1}{L_{\boldsymbol{T}_{\gamma^{\otimes n}}^{(d)}}} - 1\right), \tag{4.1}$$

there are at least $(|\gamma|^{(d-1)})^h$ vertex-disjoint paths originating in $B_G(A,R)$ and leaving $B_G(A,R')$.

Finally, note that, by assumption, we have $\min(\gamma) \geqslant b$, and therefore $\min(\gamma^{\otimes n}) \geqslant b^n$, implying that

$$L_{\boldsymbol{T}_{\gamma^{\otimes n}}^{(d)}} = b^{-n}.$$

Since $\operatorname{diam}(G) \leqslant (d+1)b^n$ by Lemma 3.7, the constraint (4.1) is implied by

$$R' < \frac{1}{2}\left(\frac{\operatorname{diam}(G)}{d+1} - 1\right).$$

We may assume that $\operatorname{diam}(G) > 6(d+1)$, otherwise the statement of the lemma is vaccuous (since we may choose the constant c sufficiently small depending on d), in which case this is implied by

$$R' \leqslant \frac{\operatorname{diam}(G)}{3(d+1)},$$

completing the proof. □

5 Sphere-packing representations

We will now prove Lemma 2.6 and Theorem 2.10. To this end, we require some regularity from our cube packings. Say that two closed, axis-parallel cubes $A, B \subseteq \mathbb{R}^d$ are *neatly tangent* if:

(i) A and B are interior-disjoint.

(ii) If A and B intersect along $(d-1)$-dimensional faces $F_A \subseteq A$ and $F_B \subseteq B$, then either $F_A \subseteq A \cap B$ or $F_B \subseteq A \cap B$.

A collection C of closed, axis-parallel cubes is *a neat cube packing* if every pair $A \neq B \in C$ is either neatly tangent or else disjoint. The *aspect ratio of the packing* C is defined by

$$\alpha(C) := \max \left\{ \frac{\ell(A)}{\ell(B)} : A, B \in C, A \cap B \neq \emptyset \right\},$$

where $\ell(A)$ denotes the sidelength of a cube A. Say that a graph G is *admits an α-uniform neat cube packing in $\mathbb{R}^d$* if G is the tangency graph of a neat cube packing C with $\alpha(C) \leqslant \alpha$.

Lemma 5.1. *If $\gamma = \langle \gamma_1, \ldots, \gamma_b \rangle$ satisfies $\gamma_1 = \gamma_b = b = \min(\gamma)$ and (2.2) holds, then the graphs $G(\boldsymbol{T}_\gamma^{(d)})$ and $\hat{G}_\gamma^{(d)}$ admit an α-uniform neat cube packing in $\mathbb{R}^d$ with $\alpha = \alpha_{\boldsymbol{T}_\gamma^{(d)}}$.*

Proof. First take $C := \tilde{\boldsymbol{T}}_\gamma^{(d)}$, as defined in Section 2.2. Under the integrality assumptions on γ, it holds that C is a neat packing, since one of the ratios γ_{i+1}/γ_i or γ_i/γ_{i+1} is an integer for every $1 \leqslant i < b$ (see Figure 3(b) for an illustration). Moreover, we have

$$\alpha(C) = \alpha_{\boldsymbol{T}_\gamma^{(d)}} = \alpha_{\tilde{\boldsymbol{T}}_\gamma^{(d)}}.$$

For the second assertion, we take $C := \boldsymbol{T}_{\gamma^{\otimes\infty}}^{(d)}$ (as defined in Section 2.2). Corollary 3.6 asserts that

$$\alpha_{\boldsymbol{T}_{\gamma^{\otimes n}}^{(d)}} \leqslant \alpha_{\boldsymbol{T}_\gamma^{(d)}}, \quad \forall n \geqslant 0,$$

hence $\alpha_{\boldsymbol{T}_{\gamma^{\otimes\infty}}^{(d)}} < \infty$, as desired. □

Given the preceding lemma, the next result suffices to prove Lemma 2.6 and Theorem 2.10.

Lemma 5.2. *If G admits an α-uniform neat cube packing in $\mathbb{R}^d$, then there are numbers $m \leqslant O(\alpha d)$ and $M \leqslant O(\alpha^2 d)$ such that the subdivision $[G]_m$ is M-uniformly sphere-packed in $\mathbb{R}^d$.*

To prove this, we need a simple result on sphere packings that satisfy prescribed tangencies. For a general closed axis-parallel cube $C \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, we define ∂C to denote the boundary of C in $\mathbb{R}^d$, and we use $\|\cdot\|$ for the standard Euclidean distance. For a point $x \in \mathbb{R}^d$ and a set $S \subseteq \mathbb{R}^d$, define $\operatorname{dist}_2(x, S) := \inf\{\|x - y\| : y \in S\}$. Let us also define

$$\partial_\varepsilon C := \{x \in \partial C : \#\{F \in \mathcal{F}_C : \operatorname{dist}_2(x, F) > \varepsilon \ell(C)\} = 2d - 1\},$$

where $\ell(C)$ is the sidelength of C, and $\mathcal{F}_C$ is the collection of the $2d$ facets of C (i.e., the $(d-1)$-dimensional faces of C). These are the boundary points that lie on exactly once face of C and are $\varepsilon\ell(C)$-far from every other.

Lemma 5.3. *For every $d \geqslant 2$ and $\varepsilon \in (0, 1/2)$, the following holds. Consider a closed axis-parallel cube $C \subseteq \mathbb{R}^d$ of sidelength ℓ, and a set of points $P \subseteq \partial_\varepsilon C$ such that $\|x - y\| \geqslant \varepsilon\ell$ for every $x \neq y \in P$. Then there is a finite collection of interior-disjoint spheres $\mathcal{S}$ contained in C such that:*

1. *The radius of every sphere in $\mathcal{S}$ is at least $\varepsilon\ell/(60d)$.*
2. *For every $p \in P$, there is a sphere $S(p) \in \mathcal{S}$ tangent to p.*
3. *The tangency graph of $\mathcal{S}$ is the m-subdivision of a star graph whose leaves are the spheres $\{S(p) : p \in P\}$, with $m = 2 + \lceil \frac{6d}{\varepsilon} \rceil$.*

Proof. By scaling and translation, it suffices to prove the lemma for $C = [0, 1]^d$. Denote $c_0 := (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, and define the sphere

$$S_0 := \{x \in \mathbb{R}^d : \|x - c_0\| = 1/4\}.$$

For each $p \in P$, define $S(p)$ to be the unique sphere of radius $\varepsilon/8$ that is contained in $[0, 1]^d$ and such that $S(p) \cap \partial[0, 1]^d = \{p\}$. Such a sphere exists because $p \in \partial_\varepsilon[0, 1]^d$.

Let c_p denote the center of $S(p)$, and let $[c_0, c_p]$ denote the line segment from c_0 to c_p. Define z_p to be the point where $[c_0, c_p]$ intersects S_0, and define z'_p to be the point where $[c_0, c_p]$ intersects $S(p)$. Let $[z_p, z'_p] \subseteq [c_0, c_p]$ denote the line segment connecting z_p to z'_p.

As $\|x - y\| \geqslant \varepsilon$ for all $x \neq y \in P$, it holds that $\|z'_x - z'_y\| \geqslant \varepsilon - 4(\varepsilon/8) \geqslant \varepsilon/2$. Note that $[0, 1]^d \subseteq B_2(c_0, \sqrt{d}/2)$, where the latter object is the Euclidean ball of radius $\sqrt{d}/2$ about c_0. Let $\tilde{z}_x$ denote the point where the line through $[c_0, c_p]$ intersects $\partial B_2(c_0, \sqrt{d}/2)$. Then we have $\|\tilde{z}_x - \tilde{z}_y\| \geqslant \|z'_x - z'_y\|$, and by similarity of the triangles defined by $\{c_0, \tilde{z}_x, \tilde{z}_y\}$ and $\{c_0, z'_x, z'_y\}$, it holds that

$$\|z'_x - z'_y\| \geqslant \frac{\|\tilde{z}_x - \tilde{z}_y\|}{\sqrt{d}} \geqslant \frac{\varepsilon}{2\sqrt{d}}.$$

It follows that

$$\min\{\|a - b\| : a \in [z_x, z'_x], b \in [z_y, z'_y]\} \geqslant \|z_x - z_y\| \geqslant \frac{\varepsilon}{2\sqrt{d}}. \tag{5.1}$$

Note also that for every $p \in P$,

$$\frac{1}{8} \leqslant \frac{1}{4} - \frac{\varepsilon}{4} \leqslant \|c_0 - c_p\| - \left(\frac{1}{4} + \frac{\varepsilon}{8}\right) \leqslant \left\|z_p - z'_p\right\| \leqslant \|c_0 - c_p\| \leqslant \operatorname{diam}_2([0, 1]^d) \leqslant \sqrt{d}, \tag{5.2}$$

where we have used $\varepsilon < 1/2$.

Let $\gamma_p : [0, \|z_p - z'_p\|] \to [z_p, z'_p]$ be a parameterization of $[z_p, z'_p]$ by arclength. Define $k := \lceil \frac{6d}{\varepsilon} \rceil$ and $r_p := \frac{\|z_p - z'_p\|}{2k}$, and let $\tilde{S}_p$ be the collecion of interior-disjoint spheres of radius r_p centered at the points

$$\gamma_p(r_p), \gamma_p(2r_p), \gamma_p(4r_p), \ldots, \gamma_p(2(k-1)r_p).$$

Note that the tangency graph of $\tilde{S}_p$ is a path, and that the first sphere is tangent to S_0, while the last is tangent to $S(p)$.

By (5.2), for each $p \in P$, we have $r_P \in \left[\frac{\varepsilon}{60d}, \frac{\varepsilon}{6\sqrt{d}}\right]$. In particular, (5.1) implies that if $S \in \tilde{S}_p$ and $S' \in \tilde{S}_{p'}$ for $p \neq p' \in P$, then S and S' are disjoint.

Thus the collection of spheres

$$\mathcal{S} := \{S_0\} \cup \{S(p) : p \in P\} \cup \bigcup_{p \in P} \tilde{S}_p$$

satisfies the conditions of the lemma. □

Proof of Lemma 5.2. Suppose that C is a neat cube packing whose tangency graph is G and such that $\alpha(C) \leqslant \alpha$. For every pair $A, B \in C$ with $A \cap B \neq \emptyset$, let $c(A, B)$ be the center of mass of $A \cap B$.

Define the set of points $P_A := \{c(A, B) : B \in C, A \cap B \neq \emptyset\}$. Since C is a neat cube packing and we have $\ell(B) \geqslant \ell(A)/\alpha$, it follows that with $\varepsilon := 1/(2\alpha)$, we have $P_A \subseteq \partial_\varepsilon A$, and

$$\|x - y\| \geqslant \varepsilon \ell(A), \quad \forall x \neq y \in P_A.$$

Therefore we can use Lemma 5.3 to replace each cube $A \in C$ by a corresponding collection $\mathcal{S}_A$ of spheres whose tangency graph is the m-subdivision of a star, the leaves of which are tangent to the points in P_A. Note that Lemma 5.3 gives $m = 2 + \lceil 12\alpha d \rceil$.

Moreover, any two adjacent spheres have their ratio of radii contained in the interval $[M^{-1}, M]$ for

$$M := \alpha \frac{60d}{\varepsilon} = 120\alpha^2 d.$$

Thus if we define $\mathcal{S} := \bigcup_{A \in C} \mathcal{S}_A$, then $[G]_{2m}$ is the tangency graph of $\mathcal{S}$, and the corresponding sphere-packing is M-uniform. □

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References

[Ang03] O. Angel. Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003. 2

[BK02] Mario Bonk and Bruce Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183, 2002. 3

[BP11] Itai Benjamini and Panos Papasoglu. Growth and isoperimetric profile of planar graphs. *Proc. Amer. Math. Soc.*, 139(11):4105–4111, 2011. 2, 3

[BS01] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13 pp., 2001. 3

[DG20] Jian Ding and Ewain Gwynne. The Fractal Dimension of Liouville Quantum Gravity: Universality, Monotonicity, and Bounds. *Comm. Math. Phys.*, 374(3):1877–1934, 2020. 3

[EL20] F. Ebrahimnejad and J. R. Lee. On planar graphs of uniform polynomial growth. To appear, *Probab. Theory Related Fields.* Preprint at arXiv:math/2005.03139, 2020. 3, 4, 9

[GHS20] Ewain Gwynne, Nina Holden, and Xin Sun. A mating-of-trees approach for graph distances in random planar maps. *Probab. Theory Related Fields*, 177(3-4):1043–1102, 2020. 2, 3

[Kri05] M. A. Krikun. Uniform infinite planar triangulation and related time-reversed critical branching process. *J. Math. Sci.*, 131:5520–5537, 2005. 2

[LT79] Richard J. Lipton and Robert Endre Tarjan. A separator theorem for planar graphs. *SIAM J. Appl. Math.*, 36(2):177–189, 1979. 1

[MTTV97] Gary L. Miller, Shang-Hua Teng, William Thurston, and Stephen A. Vavasis. Separators for sphere-packings and nearest neighbor graphs. *J. ACM*, 44(1):1–29, 1997. 1

[MTTV98] Gary L. Miller, Shang-Hua Teng, William Thurston, and Stephen A. Vavasis. Geometric separators for finite-element meshes. *SIAM J. Sci. Comput.*, 19(2):364–386, 1998. 1

[PRS94] Serge Plotkin, Satish Rao, and Warren D. Smith. Shallow excluded minors and improved graph decompositions. In *Proceedings of the Fifth Annual ACM-SIAM Symposium on Discrete Algorithms (Arlington, VA, 1994)*, pages 462–470, New York, 1994. ACM. 2

[ST07] Daniel A. Spielman and Shang-Hua Teng. Spectral partitioning works: Planar graphs and finite element meshes. *Linear Algebra and its Applications: Special Issue in honor of Miroslav Fiedler*, 421(2–3):284–305, March 2007. 1

[Ten98] Shang-Hua Teng. Combinatorial aspects of geometric graphs. *Computational Geometry*, 9(4):277–287, 1998. 2